

Regular singular differential equations

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Translator's note.

This text is one of a series of translations of various papers into English. The translator takes full responsibility for any errors introduced in the passage from one language to another, and claims no rights to any of the mathematical content herein.*

What follows is a translation of the French book:

DELIGNE, P. *Equations Différentielles à Points Singuliers Réguliers*. Springer-Verlag, Lecture Notes in Mathematics **163** (1970). <https://publications.ias.edu/node/355>

We have also made changes following the errata, which was written in April 1971, by P. Deligne, at Warwick University.

*<https://thosgood.com/translations/>

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Chapter 0

Introduction

(**!TODO! errata**)

If X is a (non-singular) complex-analytic manifold, then there is an equivalence between the notions of

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- a) local systems of complex vectors on X ; and
- b) vector bundles on X endowed with an integrable connection.

The latter of these two notions can be adapted in an evident way to the case where X is a non-singular algebraic variety over a field k (which we will take here to be of characteristic 0). However, general algebraic vector bundles with integrable connections are pathological (see ??); we only obtain a reasonable theory if we impose a “regularity” condition at infinity. By a theorem of Griffiths [8], this condition is automatically satisfied for “Gauss-Manin connections” (see ??). In dimension one, this is closely linked to the idea of regular singular points of a differential equation (see §(I.4) and §(II.1)).

In Chapter I, we explain the different forms that the notion of an integrable connection can take. In Chapter II, we prove the fundamental facts concerning regular connections. In Chapter III, we translate certain results that we have obtained into the language of Nilsson class functions, and, as an application of the regularity theorem (?), we explain the proof by Brieskorn [5] of the monodromy theorem.

These notes came from the non-crystalline part of a seminar given at Harvard during the autumn of 1969, under the title: “Regular singular differential equations and crystalline cohomology”.

I thank the assistants of this seminar, who had to be subjected to often unclear exposés, and who allowed me to bring numerous simplifications.

I also thank N. Katz, with whom I had numerous and useful conversations, and to whom are due the principal results of section §(II.1).

Notation and terminology

Within a single chapter, the references follow the decimal system. A reference to a different chapter (resp. to the current introduction) is preceded by the Roman numeral of the chapter (resp. by 0).

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We will use the following definitions:

- (0.1) *analytic space*: the analytic spaces are complex and of locally-finite dimension. They are assumed to be σ -compact, but not necessarily separated.
- (0.2) *multiform function*: a synonym for multivalued function — for a precise definition, see (6.2).
- (0.3) *immersion*: following the tradition of algebraic geometers, immersion is a synonym for “embedding”.
- (0.4) *smooth*: a morphism $f: X \rightarrow S$ of analytic spaces is smooth if, locally on X , it is isomorphic to the projection from $D^n \times S$ to S , where D^n is an open polydisc.
- (0.5) *locally paracompact*: a topological space is locally paracompact if every point has a paracompact neighbourhood (and thus a fundamental system of paracompact neighbourhoods).
- (0.6) non-singular (or smooth) *complex algebraic variety*: a smooth scheme of finite type over $\text{Spec}(\mathbb{C})$.
- (0.7) (complex) *analytic manifold*: a non-singular (or smooth) analytic space.
- (0.8) *covering*: following the tradition of topologists, a covering is a continuous map $f: X \rightarrow Y$ such that every point $y \in Y$ has a neighbourhood V such that $f|_V$ is isomorphic to the projection from $F \times V$ to V , where F is discrete.

Chapter I

Dictionary

In this chapter, we explain the relations between various aspects and various uses of the notion of “local systems of complex vectors”. The equivalence between the points of view considered has been well known for a long time.

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We do not consider the “crystalline” point of view; see [4, 10].

I.1 Local systems and the fundamental group

Definition 1.1. Let X be a topological space. A *complex local system* on X is a sheaf of complex vectors on X that, locally on X , is isomorphic to a constant sheaf \mathbb{C}^n ($n \in \mathbb{N}$).

1.2. Let X be a locally path-connected and locally simply path-connected topological space, along with a basepoint $x_0 \in X$. To avoid any ambiguity, we point out that:

- a) The fundamental group $\pi_1(X, x_0)$ of X at x_0 has elements given by homotopy classes of loops based at x_0 ;
- b) If $\alpha, \beta \in \pi_1(X, x_0)$ are represented by loops a and b , then $\alpha\beta$ is represented by the loop ab obtained by juxtaposing b and a , in that order.

Let \mathcal{F} be a locally constant sheaf on X . For every path $a: [0, 1] \rightarrow X$, the inverse image $a^*\mathcal{F}$ of \mathcal{F} on $[0, 1]$ is a locally constant, and thus constant, sheaf, and there exists exactly one isomorphism between $a^*\mathcal{F}$ and the constant sheaf defined by the set $(a^*\mathcal{F})_0 = \mathcal{F}_{a(0)}$. This isomorphism defines an isomorphism $a(\mathcal{F})$ between $(a^*\mathcal{F})_0$ and $(a^*\mathcal{F})_1$, i.e. an isomorphism

$$a(\mathcal{F}): \mathcal{F}_{a(0)} \rightarrow \mathcal{F}_{a(1)}.$$

This isomorphism depends only on the homotopy class of a , and satisfies $ab(\mathcal{F}) = a(\mathcal{F}) \cdot b(\mathcal{F})$. In particular, $\pi_1(X, x_0)$ acts (on the left) on the fibre \mathcal{F}_{x_0} of \mathcal{F} at x_0 . It is well known that:

Proposition 1.3. Under the hypotheses of (1.2), with X connected, the functor $\mathcal{F} \mapsto \mathcal{F}_{x_0}$ is an equivalence between the category of locally constant sheaves on X and the category of sets endowed with an action by the group $\pi_1(X, x_0)$.

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Corollary 1.4. *Under the hypotheses of (1.2), with X connected, the functor $\mathcal{F} \mapsto \mathcal{F}_{x_0}$ is an equivalence between the category of complex local systems on X and the category of complex finite-dimensional representations of $\pi_1(X, x_0)$.*

1.5. Under the hypotheses of (1.2), if $a: [0, 1] \rightarrow X$ is a path, and b a loop based at $a(0)$, then $aba^{-1} = a(b)$ is a path based at $a(1)$. Its homotopy class depends only on the homotopy classes of a and b . This construction defines an isomorphism between $\pi_1(X, a(0))$ and $\pi_1(X, a(1))$.

Proposition 1.6. *Under the hypotheses of (1.5), there exists, up to unique isomorphism, exactly one locally constant sheaf of groups $\Pi_1(X)$ on X (the fundamental groupoid), endowed, for all $x_0 \in X$, with an isomorphism*

$$\Pi_1(X)_{x_0} \simeq \pi_1(X, x_0) \tag{1.6.1}$$

and such that, for every path $a: [0, 1] \rightarrow X$, the isomorphism in (1.5) between $\pi_1(X, a(0))$ and $\pi_1(X, a(1))$ can be identified, via (1.6.1), with the isomorphism in (1.2) between $\Pi_1(X)_{a(0)}$ and $\Pi_1(X)_{a(1)}$.

If X is connected, with base point x_0 , then the sheaf $\Pi_1(X)$ corresponds, via the equivalence in (1.3), to the group $\pi_1(X, x_0)$ endowed with its action over itself by inner automorphisms.

Proposition 1.7. *If \mathcal{F} is a locally constant sheaf on X , then there exists exactly one action (said to be canonical) of $\Pi_1(X)$ on \mathcal{F} that, at each $x_0 \in X$, induces the action from (1.2) of $\pi_1(X, x_0)$ on \mathcal{F} .*

I.2 Integrable connections and local systems

2.1. Let X be an analytic space (0.1). We define a (holomorphic) *vector bundle* on X to be a locally free sheaf of modules that is of finite type over the structure sheaf \mathcal{O} of X . If \mathcal{V} is a vector bundle on X , and x a point of X , then we denote by $\mathcal{V}_{(x)}$ the free $\mathcal{O}_{(x)}$ -module of finite type of germs of sections of \mathcal{V} . If \mathfrak{m}_x is the maximal ideal of $\mathcal{O}_{(x)}$, then we define the *fibre at x of the vector bundle \mathcal{V}* to be the **!TODO!** of finite rank

$$\mathcal{V}_x = \mathcal{V}_{(x)} \otimes_{\mathcal{O}_{(x)}} \mathcal{O}_{(x)}/\mathfrak{m}_x. \tag{2.1.1}$$

If $f: X \rightarrow Y$ is a morphism of analytic spaces, then the *inverse image* of a vector bundle \mathcal{V} on Y is the vector bundle $f^*\mathcal{V}$ on X given by the inverse image of \mathcal{V} as a coherent module: if $f^*\mathcal{V}$ is the sheaf-theoretic inverse image of \mathcal{V} , then

$$f^*\mathcal{V} \simeq \mathcal{O}_X \otimes_{f^*\mathcal{O}_Y} f^*\mathcal{V} \tag{2.1.2}$$

In particular, if $x: P \rightarrow X$ is the morphism from the point space P to X defined by a point x of X , then

$$\mathcal{V}_x \simeq x^*\mathcal{V}. \tag{2.1.3}$$

2.2. Let X be a complex-analytic manifold (0.7) and \mathcal{V} a vector bundle on X . The elders would have defined a (holomorphic) connection on \mathcal{V} as the data, for every pair of points

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(x, y) that are first order infinitesimal neighbours in X , of an isomorphism $\gamma_{y,x}: \mathcal{V}_x \rightarrow \mathcal{V}_y$ that depends holomorphically on (x, y) and is such that $\gamma_{x,x} = \text{Id}$.

Suitably interpreted, this “definition” coincides with the currently fashionable definition (2.2.4) given below (which we not be use in the rest of the section).

It suffices to understand “point” to mean “point with values in any analytic space”:

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2.2.1. A point in an analytic space X with values in an analytic space S is a morphism from S to X .

2.2.2. If Y is a subspace of X , then the n th infinitesimal neighbourhood of Y in X is the subspace of X defined locally by the $(n + 1)$ th power of the ideal of \mathcal{O}_X that defines Y .

2.2.3. Two points $x, y \in X$ with values in S are said to be first order infinitesimal neighbours if the map $(x, y): S \rightarrow X \times X$ that they define factors through the first order infinitesimal neighbourhood of the diagonal of $X \times X$.

2.2.4. If X is a complex-analytic manifold and \mathcal{V} is a vector bundle on X , then a (holomorphic) connection γ on \mathcal{V} consists of the following data:

for every pair (x, y) of points of X with values in an arbitrary analytic space S , with x and y first order infinitesimal neighbours, an isomorphism $\gamma_{x,y}: x^*\mathcal{V} \rightarrow y^*\mathcal{V}$; this data is subject to the conditions:

(i) (functoriality) For any $f: T \rightarrow S$ and any first order infinitesimal neighbours $x, y: S \rightrightarrows X$, we have $f^*(\gamma_{y,x}) = \gamma_{yf,xf}$.

(ii) We have $\gamma_{x,x} = \text{Id}$.

2.3. Let X_1 be the first-order infinitesimal neighbourhood of the diagonal X_0 of $X \times X$, and let p_1 and p_2 be the two projections of X_1 to X . By definition, the vector bundle $P^1(\mathcal{V})$ of first-order jets of sections of \mathcal{V} is the bundle $(p_1)_*p_2^*\mathcal{V}$. We denote by j^1 the first-order differential operator that sends each section of \mathcal{V} to its first-order jet:

$$j^1: \mathcal{V} \rightarrow P^1(\mathcal{V}) \simeq \mathcal{O}_{X_1} \otimes_{\mathcal{O}_X} \mathcal{V}.$$

A connection ((2.2.4)) can be understood as a homomorphism (which is automatically an isomorphism)

$$\gamma = p_1^*\mathcal{V} \rightarrow p_2^*\mathcal{V} \tag{2.3.1}$$

which induces the identity over X_0 . Since

$$\text{Hom}_{X_1}(p_1^*\mathcal{V}, p_2^*\mathcal{V}) \simeq \text{Hom}(\mathcal{V}, (p_1)_*p_2^*\mathcal{V}),$$

a connection can also be understood as a (\mathcal{O} -linear) homomorphism

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$$D: \mathcal{V} \rightarrow P^1(\mathcal{V}) \tag{2.3.2}$$

such that the obvious composite arrow

$$\mathcal{V} \xrightarrow{D} P^1(\mathcal{V}) \rightarrow \mathcal{V}$$

is the identity. The sections Ds and $j^1(s)$ of $P^1(\mathcal{V})$ thus have the same image in \mathcal{V} , and $j^1(s) - D(s)$ can be identified with a section ∇s of $\Omega_X^1 \otimes \mathcal{V} \simeq \text{Ker}(P^1(\mathcal{V}) \rightarrow \mathcal{V})$:

$$\nabla: \mathcal{V} \rightarrow \Omega^1(X) \tag{2.3.3}$$

$$j^1(s) = D(s) + \nabla s. \quad (2.3.4)$$

In other words, a connection (2.2.4), allowing us to compare two neighbouring fibres of \mathcal{V} , also allows us to define the differential ∇s of a section of \mathcal{V} .

Conversely, equation (2.3.4) allows us to define D , and thus γ , from the covariant derivative ∇ . For D to be linear, it is necessary and sufficient for ∇ to satisfy the identity

$$\nabla(fs) = df \cdot s + f \cdot \nabla s \quad (2.3.5)$$

Definition (2.2.4) is thus equivalent to the following definition, due to J.L. Koszul.

Definition 2.4. Let \mathcal{V} be a (holomorphic) vector bundle on a complex-analytic manifold X . A *holomorphic connection* (or simply, *connection*) on \mathcal{V} is a \mathbb{C} -linear homomorphism

$$\nabla: \mathcal{V} \rightarrow \Omega_X^1(\mathcal{V}) = \Omega_X^1 \otimes_{\mathcal{O}} \mathcal{V}$$

that satisfies the Leibniz identity ((2.3.5)) for local sections f of \mathcal{O} and s of \mathcal{V} . We call ∇ the *covariant derivative* defined by the connection.

2.5. If the vector bundle \mathcal{V} is endowed with a connection Γ with covariant derivative ∇ , and if w is a holomorphic vector field on X , then we set, for every local section v of \mathcal{V} over an open subset U of X ,

$$\nabla_w(v) = \langle \nabla v, w \rangle \in \mathcal{V}(U).$$

We call $\nabla_w: \mathcal{V} \rightarrow \mathcal{V}$ the *covariant derivative along the vector field w* .

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2.6. If ${}_1\Gamma$ and ${}_2\Gamma$ are connections on X , with covariant derivatives ${}_1\nabla$ and ${}_2\nabla$ (respectively), then ${}_2\nabla - {}_1\nabla$ is a \mathcal{O} -linear homomorphism from \mathcal{V} to $\Omega_X^1(\mathcal{V})$. Conversely, the sum of ${}_1\nabla$ and such a homomorphism defines a connection on \mathcal{V} . Thus connections on \mathcal{V} form a principal homogeneous space (or torsor) on $\underline{\text{Hom}}(\mathcal{V}, \Omega_X^1(\mathcal{V})) \simeq \Omega_X^1(\underline{\text{End}}(\mathcal{V}))$.

2.7. If vector bundles are endowed with connections, then every vector bundle obtained by a “tensor operation” is again endowed with a connection. This is evident with (2.2.4). More precisely, let \mathcal{V}_1 and \mathcal{V}_2 be vector bundles endowed with connections with covariant derivatives ∇_1 and ∇_2 .

2.7.1. We define a connection on $\mathcal{V}_1 \oplus \mathcal{V}_2$ by the formula

$$\nabla_w(v_1 + v_2) = {}_1\nabla_w(v_1) + {}_2\nabla_w(v_2)$$

2.7.2. We define a connection on $\mathcal{V}_1 \otimes \mathcal{V}_2$ by the Leibniz formula

$$\nabla_w(v_1 \otimes v_2) = \nabla_w v_1 \cdot v_2 + v_1 \cdot \nabla_w v_2.$$

2.7.3. We define a connection on $\underline{\text{Hom}}(\mathcal{V}_1, \mathcal{V}_2)$ by the formula

$$(\nabla_w f)(v_1) = {}_2\nabla_2(f(v_1)) - f({}_1\nabla_1 v_1).$$

The canonical connection on \mathcal{O} is the connection for which $\nabla f = df$.

Let \mathcal{V} be a vector bundle endowed with a connection.

2.7.4. We define a connection on the dual \mathcal{V}^\vee of \mathcal{V} via (2.7.3) and the defining isomorphism $\mathcal{V}^\vee = \underline{\text{Hom}}(\mathcal{V}, \mathcal{O})$. We have

$$\langle \nabla_w v', v \rangle = \partial_w \langle v', v \rangle - \langle v', \nabla_w v \rangle.$$

We leave it to the reader to verify that these formulas do indeed define connections. For (2.7.2), for example, one must verify that, firstly, the given formula defines a \mathbb{C} -bilinear map from $(\mathcal{V}_1 \otimes \mathcal{V}_2)$, which means that the right-hand side $\text{II}(v_1, v_2)$ is \mathbb{C} -bilinear and such that $\text{II}(f v_1, v_2) = \text{II}(v_1, f v_2)$; secondly, one must also verify identity (2.3.5).

2.8. An \mathcal{O} -homomorphism f between vector bundles \mathcal{V}_1 and \mathcal{V}_2 endowed with connections is said to be *compatible with the connections* if

$${}_2\nabla \cdot f = f \cdot {}_1\nabla.$$

By (2.7.3), this reduces to saying that $\nabla f = 0$, if f is thought of as a section of $\underline{\text{Hom}}(\mathcal{V}_1, \mathcal{V}_2)$. For example, by (2.7.3), the canonical map

$$\text{Hom}(\mathcal{V}_1, \mathcal{V}_2) \otimes \mathcal{V}_1 \rightarrow \mathcal{V}_2$$

is compatible with the connections.

2.9. A local section v of \mathcal{V} is said to be *horizontal* if $\nabla v = 0$. If f is a homomorphism between bundles \mathcal{V}_1 and \mathcal{V}_2 endowed with connections, then it is equivalent to say either that f is horizontal, or that f is compatible with the connections (2.8).

2.10. Let \mathcal{V} be a holomorphic vector bundle on X . Define $\Omega_X^p = \wedge^p \Omega_X^1$ and $\Omega_X^p(\mathcal{V}) = \Omega_X^p \otimes_{\mathcal{O}} \mathcal{V}$ (the sheaf of *exterior differential p -forms with values in \mathcal{V}*). Suppose that \mathcal{V} is endowed with a holomorphic connection. We then define \mathbb{C} -linear morphisms

$$\nabla: \Omega_X^p(\mathcal{V}) \rightarrow \Omega_X^{p+1}(\mathcal{V}) \quad (2.10.1)$$

characterised by the following formula:

$$\nabla(\alpha, v) = d\alpha \cdot v + (-1)^p \alpha \wedge \nabla v, \quad (2.10.2)$$

where α is any local section of Ω^p , v is any local section of \mathcal{V} , and d is the exterior differential. To prove that the right-hand side $\text{II}(\alpha, v)$ of (2.10.2) defines a homomorphism (2.10.1), it suffices to show that $\text{II}(\alpha, v)$ is \mathbb{C} -bilinear and satisfies

$$\text{II}(f\alpha, v) = \text{II}(\alpha, f v).$$

But we have that

$$\begin{aligned} \text{II}(f\alpha, v) &= d(f\alpha)v + (-1)^p f\alpha \wedge \nabla v \\ &= d\alpha \cdot f v + df \wedge \alpha v + (-1)^p f\alpha \wedge \nabla v \\ &= d\alpha \cdot f v + (-1)^p \alpha \wedge (f\nabla v + df \cdot v) \\ &= \text{II}(\alpha, f v). \end{aligned}$$

Let \mathcal{V}_1 and \mathcal{V}_2 be vector bundles endowed with connections, and let \mathcal{V} be their tensor product (2.7.2). We denote by \wedge the evident maps

$$\wedge: \Omega^p(\mathcal{V}_1) \otimes \Omega^1(\mathcal{V}_2) \rightarrow \Omega^{p+q}(\mathcal{V})$$

such that, for any local section α (resp. β , resp. v_1 , resp. v_2) of Ω^p (resp. Ω^q , resp. \mathcal{V}_1 , resp. \mathcal{V}_2), we have that $(\alpha \otimes v_1) \wedge (\beta \otimes v_2) = (\alpha \wedge \beta) \otimes (v_1 \otimes v_2)$. If v_1 (resp. v_2) is any local section of $\Omega^p(\mathcal{V}_1)$ (resp. $\Omega^q(\mathcal{V}_2)$), then | p. 10

$$\nabla(v_1 \wedge v_2) = v_1 \wedge v_2 + (-1)^p v_1 \wedge v_2. \quad (2.10.3)$$

Indeed, if $v_1 = \alpha v_1$ and $v_2 = \beta v_2$, then

$$\begin{aligned} \nabla(v_1 \wedge v_2) &= \nabla(\alpha \wedge \beta \otimes v_1 \otimes v_2) \\ &= d(\alpha \wedge \beta) v_1 \otimes v_2 + (-1)^{p+q} \alpha \wedge \beta \wedge \nabla(v_1 \otimes v_2) \\ &= d\alpha \wedge \beta v_1 \otimes v_2 + (-1)^p \alpha \wedge d\beta v_1 \otimes v_2 \\ &\quad + (-1)^{p+q} \alpha \wedge \beta \wedge \nabla v_1 \otimes v_2 + (-1)^{p+q} \alpha \wedge \beta v_1 \wedge \nabla v_2 \\ &= d\alpha v_1 \wedge v_2 + (-1)^p v_1 \wedge d\beta v_2 + (-1)^p \alpha \wedge \nabla v_1 \wedge v_2 \\ &\quad + (-1)^{p+q} v_2 \wedge \beta \wedge \nabla v_2 \\ &= (d\alpha v_1 + (-1)^p \alpha \wedge \nabla v_1) \wedge v_2 + (-1)^p v_1 \wedge (d\beta v_2 + (-1)^q \beta \wedge \nabla v_2) \\ &= \nabla v_1 \wedge v_2 + (-1)^p v_1 \wedge \nabla v_2. \end{aligned}$$

Let \mathcal{V} be a vector bundle endowed with a connection. If we apply the above formula to \mathcal{O} and \mathcal{V} , then, for any local section α (resp. v) of Ω^p (resp. $\Omega^q(\mathcal{V})$), we have that

$$\nabla(\alpha \wedge v) = d\alpha \wedge v + (-1)^p \alpha \wedge \nabla v. \quad (2.10.4)$$

Iterating this formula gives

$$\begin{aligned} \nabla \nabla(\alpha \wedge v) &= \nabla(d\alpha \wedge v + (-1)^p \alpha \wedge \nabla v) \\ &= d\alpha \wedge v + (-1)^{p+1} d\alpha \wedge \nabla v + (-1)^p d\alpha \wedge \nabla v + \alpha \wedge \nabla \nabla v \\ &= \alpha \wedge \nabla \nabla v. \end{aligned} \quad (2.10.5)$$

Definition 2.11. Under the hypotheses of (2.10), the *curvature* R of the given connection on \mathcal{V} is the composite homomorphism

$$R: \mathcal{V} \rightarrow \Omega_X^2(\mathcal{V})$$

considered as a section of $\text{Hom}(\mathcal{V}, \Omega_X^2(\mathcal{V})) \simeq \Omega_X^2(\text{End}(\mathcal{V}))$.

2.12. Taking $q = 0$ in (2.10.4) gives

$$\nabla \nabla(\alpha v) = \alpha \wedge R(v), \quad (2.12.1)$$

which we write as

$$\nabla \nabla(v) = R \wedge v \quad (\text{the Ricci identity}). \quad (2.12.2)$$

We endow $\underline{\text{End}}(\mathcal{V})$ with the connection given in (2.7.3). The equation $\nabla(\nabla \nabla) = (\nabla \nabla) \nabla$ can be written as $\nabla(R \wedge v) = R \wedge \nabla v$. By (2.7.3), we have that $\nabla R \wedge v = \nabla(R \wedge v) - R \wedge \nabla v$, so that

$$\nabla R = 0 \quad (\text{the Bianchi identity}). \quad (2.12.3)$$

2.13. If α is an exterior differential p -form, then we know that

$$\begin{aligned} \langle d\alpha, X_0 \wedge \dots \wedge X_p \rangle &= \sum_i (-1)^i j_{X_i} \langle \alpha, X_0 \wedge \dots \wedge \widehat{X}_i \wedge \dots \wedge X_p \rangle \\ &\quad + \sum_{i < j} (-1)^{i+j} \langle \alpha, [X_i, X_j] \wedge X_0 \wedge \dots \wedge \widehat{X}_i \wedge \dots \wedge \widehat{X}_j \wedge \dots \wedge X_p \rangle. \end{aligned}$$

From this formula, and from (2.10.2), we see that, for any local section v of $\Omega_X^p(\mathcal{V})$, and holomorphic vector fields X_0, \dots, X_p ,

$$\begin{aligned} \langle \nabla v, X_0 \wedge \dots \wedge X_p \rangle &= \sum_i (-1)^i \nabla_{X_i} \langle v, X_0 \wedge \dots \wedge \widehat{X}_i \wedge \dots \wedge X_p \rangle \\ &\quad + \sum_{i < j} (-1)^{i+j} \langle v, [X_i, X_j] \wedge X_0 \wedge \dots \wedge \widehat{X}_i \wedge \dots \wedge \widehat{X}_j \wedge \dots \wedge X_p \rangle. \end{aligned} \quad (2.13.1)$$

In particular, for any local section v of \mathcal{V} , we have that

$$\langle \nabla \nabla v, X_1 \wedge X_2 \rangle = \nabla_{X_1} \langle \nabla v, X_2 \rangle - \nabla_{X_2} \langle v, X_1 \rangle - \langle \nabla v, [X_1, X_2] \rangle.$$

That is,

$$R(X_1, X_2)(v) = \nabla_{X_1} \nabla_{X_2} v - \nabla_{X_2} \nabla_{X_1} v - \nabla_{[X_1, X_2]} v. \quad (2.13.2)$$

Definition 2.14. A connection is said to be *integrable* if its curvature is zero, i.e. (2.13.2) if the following holds identically:

$$\nabla_{[X, Y]} = [\nabla_X, \nabla_Y].$$

If $\dim(X) \leq 1$, then every connection is integrable.

If Γ is an integrable connection on \mathcal{V} , then the morphism ∇ of (2.10.1) satisfy $\nabla \nabla = 0$, and so the $\Omega^p(\mathcal{V})$ give a differential complex $\Omega^*(\mathcal{V})$.

Definition 2.15. Under the above hypotheses, the complex $\Omega^*(\mathcal{V})$ is called the *holomorphic de Rham complex* with values in \mathcal{V} .

The results (2.16) to (2.19) that follow will be proven in a more general setting in (2.23). | p. 12

Proposition 2.16. Let V be a local complex system on a complex-analytic variety X (0.6), and let $\mathcal{V} = \mathcal{O} \otimes_{\mathbb{C}} V$.

- (i) There exists, on the vector bundle \mathcal{V} , exactly one connection (said to be *canonical*) whose horizontal sections are the local sections of the subsheaf V of \mathcal{V} .
- (ii) The canonical connection on \mathcal{V} is integrable.
- (iii) For any local section f (resp. v) of \mathcal{O} (resp. V),

$$\nabla(fv) = df \cdot v. \quad (2.16.1)$$

Proof. If ∇ satisfies (i), then (2.16.1) is a particular case of (2.3.5). Conversely, the right-hand side $\text{II}(f, v)$ of (2.16.1) is \mathbb{C} -bilinear, and thus extends uniquely to a \mathbb{C} -linear map $\nabla: \mathcal{V} \rightarrow \Omega^1(\mathcal{V})$, which we can show defines a connection. Claim (ii) is local on X , which allows us to reduce to the case where $V = \underline{\mathbb{C}}$. Then $\mathcal{V} = \mathcal{O}$, $\nabla = d$, and $\nabla_{[X, Y]} = [\nabla_X, \nabla_Y]$ by the definition of $[X, Y]$. \square

It is well known that:

Theorem 2.17. *Let X be a complex-analytic variety. Then the following functors are quasi-inverse to one another, and thus give an equivalence between the category of complex local systems on X and the category of holomorphic vector bundles with on X with integrable connections (with the morphisms being the horizontal morphisms of vector bundles):*

- a) *the complex local system V is sent to $\mathcal{V} = \mathcal{O} \otimes V$ endowed with its canonical connection;*
- b) *the holomorphic vector bundle \mathcal{V} endowed with its integrable connection is sent to the subsheaf V of \mathcal{V} consisting of horizontal sections (i.e. those v such that $\nabla v = 0$).*

These equivalences are compatible with taking the tensor product, the internal Hom, and the dual; to the unit complex local system $\underline{\mathbb{C}}$ corresponds the bundle \mathcal{O} endowed with the connection ∇ such that $\nabla f = df$.

Definition (2.10.2) implies the following:

Proposition 2.18. *If V is a complex local system on X , and if $\mathcal{V} = \mathcal{O} \otimes_{\mathbb{C}} V$, then the system of isomorphisms* | p. 13

$$\Omega_X^p \otimes_{\mathbb{C}} V \simeq \Omega_X^p \otimes_{\mathcal{O}} \mathcal{O} \otimes_{\mathbb{C}} V \simeq \Omega_X^p \otimes_{\mathcal{O}} \mathcal{V}$$

is an isomorphism of complexes

$$\Omega_X^{\bullet} \otimes_{\mathbb{C}} V \rightarrow \Omega_X^{\bullet}(\mathcal{V}).$$

From this, the holomorphic Poincaré lemma gives the following:

Proposition 2.19. *Under the hypotheses of (2.16), the complex $\Omega_X^{\bullet}(\mathcal{V})$ is a resolution of the sheaf \mathcal{V} .*

2.20. Variants.

2.20.1. If X is a differentiable manifold, and we consider C^{∞} connections on C^{∞} vector bundles, then all of the above results still hold true, mutatis mutandis. We will not use this fact.

2.20.2. Theorem (2.17) makes essential use of the non-singularity of X ; it is thus unimportant to note that this hypothesis has not been used in an essential way before (2.17)

2.20.3. The definition (2.4) of a connection and the definition (2.11) of an integrable connection are formal enough that we can transport them to the category of schemes, or in relative settings:

Definition 2.21. (i) Let $f: X \rightarrow S$ be a smooth morphism of schemes, and \mathcal{V} a quasi-coherent sheaf on X . A *relative connection* on \mathcal{V} is an $f^* \mathcal{O}_S$ -linear sheaf morphism

$$\nabla: \mathcal{V} \rightarrow \Omega_{X/S}^1(\mathcal{V})$$

(called the *covariant derivative* defined by the connection) that identically satisfies, for any local section f (resp. v) of \mathcal{O}_X (resp. \mathcal{V}),

$$\nabla(fv) = f \cdot \nabla v + df \cdot v.$$

- (ii) Given \mathcal{V} endowed with a relative connection, there exists exactly one system of $f^* \mathcal{O}_S$ -homomorphisms of sheaves

$$\nabla^{(p)}: \Omega_{X/S}^p(\mathcal{V}) \rightarrow \Omega_{X/S}^{p+1}(\mathcal{V})$$

that satisfies (2.10.4) and is such that $\nabla^{(0)} = \nabla$.

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- (iii) The *curvature* of a connection is defined by

$$R = \nabla^{(1)} \nabla^{(0)} \in \underline{\mathbf{Hom}}(\mathcal{V}, \Omega_{X/S}^2(\mathcal{V})) \cong \Omega_{X/S}^2(\underline{\mathbf{End}}(\mathcal{V})).$$

The curvature satisfies the Ricci identity (2.12.2) and the Bianchi identity (2.12.3).

- (iv) An *integrable connection* is a connection with zero curvature.

- (v) The *de Rham complex* defined by an integrable connection is the complex $(\Omega_{X/S}^p(\mathcal{V}), \nabla)$.

2.22. Let $f: X \rightarrow S$ be a *smooth* morphism of complex-analytic spaces; by hypothesis, f is thus locally (in the domain) isomorphic to a projection $\mathrm{pr}_2: \mathbb{C}^n \times S \rightarrow S$ (for some $n \in \mathbb{N}$). A *local relative system* on X is a sheaf of $f^* \mathcal{O}_S$ -modules that is locally isomorphic to the sheaf-theoretic inverse image of a coherent analytic sheaf on S . If \mathcal{V} is a coherent analytic sheaf on X , then a *relative connection* on \mathcal{V} is an $f^* \mathcal{O}_S$ -linear homomorphism

$$\nabla: \mathcal{V} \rightarrow \Omega_{X/S}^1(\mathcal{V})$$

that identically satisfies, for any local section f (resp. v) of \mathcal{O} (resp. \mathcal{V}),

$$\nabla(fv) = f \cdot \nabla v + df \cdot v.$$

A *morphism* between vector bundles endowed with relative connections is a morphism of vector bundles that commutes with ∇ . We define, as in (2.11) and (2.21), the *curvature* $R \in \Omega_{X/S}^2(\underline{\mathbf{End}}(\mathcal{V}))$ of a relative connection. A relative connection is said to be *integrable* if $R = 0$, in which case we have the *relative de Rham complex with values in \mathcal{V}* , denoted by $\Omega_{X/S}^\bullet(\mathcal{V})$, and defined as in (2.15) and (2.21).

The “absolute” statements (2.17), (2.18) and (2.19) have “relative” (i.e. “with parameters”) analogues:

Theorem 2.23. *Under the hypotheses of (2.22), we have the following.*

- (i) *For every relative local system V on X , there exists a coherent analytic sheaf $\mathcal{V} = \mathcal{O}_X \otimes_{f^* \mathcal{O}_S} V$, and exactly one relative connection, said to be *canonical*, such that a local section v of \mathcal{V} is *horizontal* (i.e. such that $\nabla v = 0$) if and only if v is a section of V ; this connection is *integrable*.*
- (ii) *Given a relative local system V on X , the de Rham complex defined by $\mathcal{V} = \mathcal{O}_X \otimes_{f^* \mathcal{O}_S} V$, endowed with its canonical connection, is a resolution of the sheaf V .*
- (iii) *The following functors are quasi-inverse to one another, and thus give an equivalence between the category of relative local systems on X and the category of coherent analytic sheaves on X endowed with a relative integrable connection:*

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- a) the relative local system V is sent to $\mathcal{V} = \mathcal{O}_X \otimes_{f^* \mathcal{O}_S} V$ endowed with its canonical connection;
- b) the coherent analytic sheaf \mathcal{V} on X endowed with a relative integrable connection is sent to the subsheaf consisting of its horizontal sections (i.e. the sections v such that $\nabla v = 0$).

Proof. We first prove (i). To show that \mathcal{V} is coherent, it suffices to do so locally, for $V = f^* V_0$, in which case \mathcal{V} is the inverse image, in the sense of coherent analytic sheaves, of \mathcal{V}_0 . The canonical relative connection necessarily satisfies, for any local section f (resp v_0) of \mathcal{O}_X (resp. V),

$$\nabla(f v_0) = df \cdot v_0. \quad (2.23.1)$$

The right-hand side $\text{II}(f, v_0)$ of this equation is biadditive in f and v_0 , and satisfies, for any local section g of $f^* \mathcal{O}_S$, the identity

$$\text{II}(f g, v_0) = \text{II}(f, g v_0)$$

(using the fact that $dg = 0$ in $\Omega_{X/S}^1$). We thus deduce the existence and uniqueness of a relative connection ∇ that satisfies (2.23.1). Finally, we have that

$$\nabla \nabla(f v_0) = \nabla(df \cdot v_0) = ddf \cdot v_0 = 0,$$

and so the canonical connection ∇ is integrable. The fact that only the sections of V are horizontal is a particular case of (ii), which is proven below. \square

2.23.2. We first of all consider the particular case of (ii) where $S = D^n$, $X = D^n \times D^m$, $f = \text{pr}_2$, and the relative local system V is the inverse image of \mathcal{O}_S . The complex of global sections

$$0 \rightarrow \Gamma(f^* \mathcal{O}_S) \rightarrow \Gamma(\mathcal{O}_X) \rightarrow \Gamma(\Omega_{X/S}^1) \rightarrow \dots$$

is acyclic, since it admits the homotopy operator H defined below. | p. 16

- a) $H: \Gamma(\mathcal{O}_X) \rightarrow \Gamma(f^* \mathcal{O}_S) = \Gamma(S, \mathcal{O}_S)$ is the inverse image under the zero section of f ;
- b) an element $\omega \in \Gamma(\Omega_{X/S}^p)$ (where $p > 0$) can be represented in a unique way as a sum of convergent series:

$$\omega = \sum_{\substack{I \subset [1, m] \\ |I|=p}} \sum_{\underline{n} \in \mathbb{N}^{m+n}} a_{\underline{n}}^I \left(\prod_{i \in I} x_i^{n_i} dx_i \right) \left(\prod_{i \in [1, m+n] \setminus I} x_i^{n_i} \right)$$

and we set

$$H(\omega) = \sum_{I \subset [1, m]} \sum_{j \in I} \sum_{\underline{n} \in \mathbb{N}^{m+n}} a_{\underline{n}}^I \left(\prod_{\substack{j \in I \\ i \neq j}} x_j^{n_j} dx_j \frac{x_j^{n_j+1}}{n_j+1} \right) \left(\prod_{i \in [1, m+n] \setminus I} x_i^{n_i} \right)$$

This remains true if we replace D^{m+n} by a smaller polycylinder, and so $\Omega_{X/S}^\bullet$ is a resolution of $f^* \mathcal{O}_S$.

2.23.3. We now prove (ii), which is of a local nature on X and S . Denoting by D the open unit disc, we can thus restrict to the case where S is a closed analytic subset of the polycylinder D^n , where $X = D^m \times S$, with $f = \text{pr}_2$, and where V is the inverse image of a coherent analytic sheaf V_0 on S . Applying the syzygy theorem, and possibly shrinking X and S , we can further suppose that the direct image of V_0 on D^n , which we again denote by V_0 , admits a finite resolution \mathcal{L} by free coherent \mathcal{O}_{D^n} -modules. To prove (ii), we are allowed to replace V_0 by its direct image on D^n , and to suppose that $D^n = S$, which we now do.

If Σ_0 is a short exact sequence of coherent \mathcal{O}_S -modules

$$\Sigma_0: 0 \rightarrow V'_0 \rightarrow V_0 \rightarrow V''_0 \rightarrow 0,$$

then let $\Sigma = f^*\Sigma_0$ be the exact sequence of relative local systems given by the inverse image of Σ_0 (which is exact since f^* is an exact functor), and let $\Omega_{X/S}^*(\Sigma)$ be the corresponding exact sequence of relative de Rham complexes:

$$\Omega_{X/S}^*(\Sigma): 0 \rightarrow \Omega_{X/S}^p \otimes_{f^*\mathcal{O}_S} f^*V'_0 \rightarrow \Omega_{X/S}^p \otimes_{f^*\mathcal{O}_S} f^*V_0 \rightarrow \Omega_{X/S}^p \otimes_{f^*\mathcal{O}_S} f^*V''_0 \rightarrow 0.$$

This sequence is exact since $\Omega_{X/S}^p$ is flat over $f^*\mathcal{O}_S$, since it is locally free over \mathcal{O}_X which is itself flat over $f^*\mathcal{O}_S$.

The snake lemma applied to $\Omega_{X/S}^*(\Sigma)$ shows that, if claim (ii) is satisfied for any two of relative local systems f^*V_0 , $f^*V'_0$, and $f^*V''_0$, then it is again satisfied for the third. We thus deduce, by induction, that, if V_0 admits a finite resolution M_\bullet by modules that satisfy (ii), then V_0 satisfies (ii). This, applied to V_0 and \mathcal{L}^* , finishes the proof of (i) and (ii). | p. 17

It follows from (ii) that the composite (iii)b◦(iii)a of the functors from (iii) is canonically isomorphic to the identity; furthermore, if V_1 and V_2 are relative local systems, and if $u: \mathcal{V}_1 \rightarrow \mathcal{V}_2$ is a homomorphism that induces 0 on V_1 , then $u = 0$, since V_1 generates \mathcal{V}_1 ; it thus follows that the functor (iii)a is fully faithful. It remains to show that every vector bundle \mathcal{V} endowed with a relative connection ∇ is given locally by a relative local system.

Case 1: $S = D^n$, $X = D^{n+1} = D^n \times D$, $f = \text{pr}_1$, and \mathcal{V} is free.

Under these hypotheses, if v is an arbitrary section of the inverse image of \mathcal{V} under the zero section s_0 of f , then there exists exactly one horizontal section \tilde{v} of \mathcal{V} that agrees with v on $s_0(S)$ (as follows from the existence and uniqueness of solutions for Cauchy problems with parameters). If $(e_i)_{1 \leq i \leq k}$ is a basis of $s_0^*\mathcal{V}$, then the \tilde{e}_i form a horizontal basis of \mathcal{V} , and (\mathcal{V}, ∇) is defined by the relative local system $f^*s_0^*\mathcal{V} \simeq f^*\mathcal{O}_S^k$.

Case 2: $S = D^n$, $X = D^{n+1} = D^n \times D$, and $f = \text{pr}_1$.

By possibly shrinking X and S , we can suppose that \mathcal{V} admits a free presentation:

$$\mathcal{V}_1 \xrightarrow{d} \mathcal{V}_0 \xrightarrow{\varepsilon} \mathcal{V} \rightarrow 0.$$

By then possibly shrinking again, we can further suppose that \mathcal{V}_0 and \mathcal{V}_1 admit connections ∇_1 and ∇_0 (respectively) such that ε and d are compatible with the connections (if (e_i) is a basis of \mathcal{V}_0 , then ∇_0 is determined by the $\nabla_0 e_i$, and it suffices to choose $\nabla_0 e_i$ such that $\varepsilon(\nabla_0 e_i) = \nabla(\varepsilon(e_i))$; we proceed similarly for ∇_1). The connections ∇_0 and ∇_1 are automatically integrable, since f is of relative

dimension 1. There thus exist (by Case 1) relative local systems V_0 and V_1 such that $(\mathcal{V}_i, \nabla_i) \simeq \mathcal{O}_X \otimes_{f^* \mathcal{O}_S} V_i$. We then have that

$$(\mathcal{V}, \nabla) \simeq \mathcal{O}_X \otimes_{f^* \mathcal{O}_S} (V_0/dV_1).$$

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Case 3: f is of relative dimension 1.

We can suppose that S is a closed analytic subset of D^n , and that $X = S \times D$ and $f = \text{pr}_1$. The relative local systems (resp. the modules with relative connections) on X can then be identified with the local relative systems (resp. the modules with relative connections) on $D^n \times D$ that are annihilated by the inverse image of the ideal that defines S , and we conclude by using Case 2.

General case. We proceed by induction on the relative dimension n of f . The case $n = 0$ is trivial. If $n \neq 0$, then we are led to the case where $X = S \times D^{n-1} \times D$ and $f = \text{pr}_1$. The bundle with connection (\mathcal{V}, ∇) induces a bundle \mathcal{V}_0 with connection on $X_0 = S \times D^{n-1} \times \{0\}$ which is, by induction, of the form $(\mathcal{V}_0, \nabla_0) = \mathcal{O}_{X_0} \otimes_{\text{pr}_1^* \mathcal{O}_S} V$. The projection f from X to $S \times D^{n-1}$ is of relative dimension 1, and the relative connection ∇ induces a relative connection for \mathcal{V} on $X/S \times D^{n-1}$. By Case 3, there exists a vector bundle V_1 on $S \times D^{n-1}$, as well as an isomorphism

$$\mathcal{V} \simeq \mathcal{O}_X \otimes_{p^* \mathcal{O}_{S \times D^{n-1}}} p^* V_1.$$

of bundles with relative connections (with respect to p).

The vector bundle V_1 can be identified with the restriction of \mathcal{V} to X_0 , whence we obtain an isomorphism

$$\alpha: \mathcal{V} \simeq \mathcal{O}_X \otimes_{f^* \mathcal{O}_S} V$$

of vector bundles, such that

- (i) the restriction of α to X_0 is horizontal ; and
- (ii) α is “relatively horizontal” with respect to p .

If v is a section of V , then condition (ii) implies that

$$\nabla_{x_n} v = 0.$$

If $1 \leq i < n$, since $R = 0$, then, by an analogous statement to (2.13.2), we have that

$$\nabla_{x_n} \nabla_{x_i} v = \nabla_{x_i} \nabla_{x_n} v = 0.$$

In other words, $\nabla_{x_i} v$ is a relative horizontal section, with respect to p , of \mathcal{V} ; by (i), it is zero on X_0 , and is thus zero, and we conclude that $\nabla v = 0$. The isomorphism α is thus horizontal, and this finishes the proof of (2.23).

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□

Some results in general topology ((2.24) to (2.27)) will be necessary to deduce (2.28) from (2.23).

Reminder 2.24. Let Y be a closed subset of a topological space X , and suppose that Y has a paracompact neighbourhood. For every sheaf \mathcal{F} on X , we have that

$$\varinjlim_{U \supset Y} H^*(U, \mathcal{F}) \simeq H^*(Y, \mathcal{F}).$$

Proof. See Godement [7, II, 4.11.1, p. 193]. \square

Corollary 2.25. Let $f: X \rightarrow S$ be a proper separated morphism between topological spaces. Suppose that S is locally paracompact (0.5). Then, for every $s \in S$, and for every sheaf \mathcal{F} on X , we have that

$$(\mathbf{R}^i f_* \mathcal{F})_s \simeq H^i(f^{-1}(s), \mathcal{F}|_{f^{-1}(s)}).$$

Proof. Since f is closed, the $f^{-1}(U)$ form a fundamental system of neighbourhoods of $f^{-1}(s)$, where the U are neighbourhoods of s . Furthermore, if U is paracompact, then $f^{-1}(U)$ is paracompact, since f is proper and separated. We conclude by (2.24). \square

Reminder 2.26. Let X be a contractible locally paracompact topological space, i an integer, and V a complex local system on X , such that $\dim_{\mathbb{C}} H^i(X, V) < \infty$. Then, for every vector space A over \mathbb{C} , possibly of infinite dimension, we have that

$$A \otimes_{\mathbb{C}} H^i(X, V) \simeq H^i(X, A \otimes_{\mathbb{C}} V). \quad (2.26.1)$$

Proof. We denote by $H_*(X, V^*)$ the singular homology of X with coefficients in V^* . The universal coefficient formula, which holds here, gives

$$H^i(X, A \otimes V) \simeq \text{Hom}_{\mathbb{C}}(H_i(X, V^*), A). \quad (2.26.2)$$

For $A = \mathbb{C}$, we thus conclude that $\dim H_i(X, V^*) < \infty$. Equation (2.26.1) then follows from (2.26.2). \square

2.27. Let $f: X \rightarrow S$ be a smooth morphism of complex-analytic spaces, and let V be a local system on X . Then the sheaf

$$V_{\text{rel}} = f^* \mathcal{O}_S \otimes_{\mathbb{C}} V \quad (2.27.1)$$

is a relative local system. We denote by $\Omega_{X/S}^{\bullet}(V)$ the corresponding de Rham complex. By (2.23), $\Omega_{X/S}^{\bullet}$ is a resolution of V_{rel} . We thus have that

$$\mathbf{R}^i f_* V_{\text{rel}} \simeq \mathbf{R}^i f_*(\Omega_{X/S}^{\bullet}(V)) \quad (2.27.2)$$

where the right-hand side is the relative hypercohomology. From (2.27.1), we thus obtain an arrow

$$\mathcal{O}_S \otimes_{\mathbb{C}} \mathbf{R}^i f_* V \rightarrow \mathbf{R}^i f_*(V_{\text{rel}}), \quad (2.27.3)$$

whence, by composition, an arrow

$$\mathcal{O}_S \otimes \mathbf{R}^i f_* V \rightarrow \mathbf{R}^i f_*(\Omega_{X/S}(V)). \quad (2.27.4)$$

Proposition 2.28. Let $f: X \rightarrow S$ be a smooth separated morphism of analytic spaces, i an integer, and V a complex local system on X . We suppose that

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a) f is topologically trivial locally on S ; and

b) the fibres of f satisfy

$$\dim H^i(f^{-1}(s), V) < \infty.$$

Then the arrow (2.27.4) is an isomorphism:

$$\mathcal{O}_S \otimes_{\mathbb{C}} \mathbf{R}^i f_* V \simeq \mathbf{R}^i f_*(\Omega_{X/S}^\bullet(V)).$$

Proof. Let $s \in S$, $Y = f^{-1}(s)$, and $V_0 = V|_Y$. To show that (2.27.4) is an isomorphism, it suffices to construct a fundamental system T of neighbourhoods of s such that the arrows

$$H^0(T, \mathcal{O}_S) \otimes H^i(T \times Y, \text{pr}_2^* V_0) \simeq H^i(T \times Y, \text{pr}_1^* \mathcal{O}_S \otimes \text{pr}_2^* V_0) \quad (2.28.1)$$

are isomorphisms. In fact, the fibre at s of (2.27.3), which is the inductive limit of the arrows (2.28.1), will then be an isomorphism.

We will prove (2.28.1) for a compact Stein neighbourhood T of s , assumed to be contractible. The arrow in (2.28.1) can then be written as

$$H^0(T, \mathcal{O}_S) \otimes H^i(Y, V_0) \simeq H^i(T \times Y, \text{pr}_1^* \mathcal{O}_S \otimes \text{pr}_2^* V_0). \quad (2.28.2)$$

We can calculate the right-hand side of (2.28.2) by using the Leray spectral sequence for $\text{pr}_2: T \times Y \rightarrow Y$. By (2.25), since $H^i(T, \mathcal{O}_S) = 0$, we have that

$$H^i(T \times Y, \text{pr}_1^* \mathcal{O}_S \otimes \text{pr}_2^* V_0) \simeq H^i(Y, H^0(T, \mathcal{O}_S) \otimes V_0),$$

and we conclude by (2.26). \square

2.29. Under the hypotheses of (2.28), with S smooth, we define the *Gauss–Manin connection* on $\mathbf{R}^i f_* \Omega_{X/S}^\bullet(V)$ as being the unique integrable connection that admits the local sections of $\mathbf{R}^i f_* V$ as its horizontal sections (2.17).

I.3 Translation in terms of first-order partial differential equations

3.1. Let X be a complex-analytic variety. If \mathcal{V} is the holomorphic vector bundle defined by a \mathbb{C} -vector space V_0 , then we have seen that \mathcal{V} admits a canonical connection with covariant derivative ${}_0\nabla$. If ∇ is the covariant derivative defined by another connection on \mathcal{V} , then we have seen (2.6) that ∇ can be written in the form

$$\nabla = {}_0\nabla + \Gamma, \quad \text{where } \Gamma \in \Omega(\underline{\text{End}}(\mathcal{V})).$$

If we identify sections of \mathcal{V} with holomorphic maps from X to V_0 , then we have that

$$\nabla v = dv + \Gamma \cdot v \quad (3.1.1)$$

If we suppose that we have chosen a basis of V , i.e. an isomorphism $e: \mathbb{C}^n \rightarrow V_0$ with coordinates (identified with basis vectors) $e_\alpha: \mathbb{C} \rightarrow V_0$, then Γ can be written as a matrix

ω_β^α of differential forms (the *matrix of forms of the connection*), and (3.1.1) can then be written as

$$(\nabla v)^\alpha = dv^\alpha + \sum_\beta \omega_\beta^\alpha v^\beta. \quad (3.1.2)$$

Let \mathcal{V} be an arbitrary holomorphic vector bundle on X . The choice of a basis $e: \mathbb{C}^n \xrightarrow{\sim} \mathcal{V}$ of \mathcal{V} allows us to think of \mathcal{V} as being defined by a constant vector **!TODO!** (\mathbb{C}^n) , and the above remarks apply: connections on \mathcal{V} correspond, via (3.1.2), with $(n \times n)$ -matrices of differential forms on X . If ω_e is the matrix of the connection ∇ in the basis e , and if $f: \mathbb{C}^n \rightarrow \mathcal{V}$ is another basis of \mathcal{V} , with matrix $A \in \text{GL}_n(\mathcal{O})$ (where $A = e^{-1}f$), then (3.1.2)

$$\begin{aligned} \nabla v &= ed(e^{-1}v) + e\omega_e e^{-1}v \\ &= fA^{-1}d(Af^{-1}v) + fA^{-1}\omega_e A f^{-1}v \\ &= fdf^{-1}v + f(A^{-1}dA + A^{-1}\omega_e A)f^{-1}v. \end{aligned}$$

Comparing this with (3.1.2) in the basis f , we find that

$$\omega_f = A^{-1}dA + A^{-1}\omega_e A. \quad (3.1.3)$$

If, further, (x^i) is a system of local coordinates on X , which defines a basis (dx^i) of Ω_X^1 , we set

$$\omega_\beta^\alpha = \sum_i \Gamma_{\beta i}^\alpha dx^i$$

and we call the holomorphic functions $\Gamma_{\beta i}^\alpha$ the *coefficients of the connection*. Equation (3.1.2) can be written as

$$(\nabla_i v)^\alpha = \partial_i v^\alpha + \sum_\beta \Gamma_{\beta i}^\alpha v^\beta. \quad (3.1.4)$$

The differential equation $\nabla v = 0$ of horizontal sections of \mathcal{V} can be written as the linear homogeneous system of first-order partial differential equations

$$\partial_i v^\alpha = - \sum_\beta \Gamma_{\beta i}^\alpha v^\beta. \quad (3.1.5)$$

3.2. With the notation of (3.1.2), and using Einstein summation notation, we have that

$$\begin{aligned} \nabla \nabla v &= \nabla((dv^\alpha + \omega_\beta^\alpha v^\beta)e_\alpha) \\ &= d(dv^\alpha + \omega_\beta^\alpha v^\beta)e_\alpha - (dv^\alpha + \omega_\beta^\alpha v^\beta) \wedge \omega_\alpha^\gamma \cdot e_\gamma \\ &= d\omega_\beta^\alpha \cdot v^\beta \cdot e_\alpha - \omega_\beta^\alpha \wedge dv^\beta \cdot e_\alpha - dv^\alpha \wedge \omega_\alpha^\gamma \cdot e_\gamma - \omega_\beta^\alpha \wedge \omega_\alpha^\gamma \cdot v^\beta e_\gamma \\ &= (d\omega_\beta^\gamma - \omega_\beta^\alpha \wedge \omega_\alpha^\gamma)v^\beta e_\gamma. \end{aligned}$$

The curvature tensor matrix is thus

$$R_\beta^\alpha = d\omega_\beta^\alpha + \sum_\gamma \omega_\gamma^\alpha \wedge \omega_\beta^\gamma, \quad (3.2.1)$$

which we can also write as

$$R = d\omega + \omega \wedge \omega. \quad (3.2.2)$$

Equation (3.2.1) gives, in a system (x^i) of local coordinates,

$$\begin{cases} R_{\beta ij}^\alpha &= (\partial_i \Gamma_{\beta j}^\alpha - \partial_j \Gamma_{\beta i}^\alpha) + (\Gamma_{\gamma i}^\alpha \Gamma_{\beta j}^\gamma - \Gamma_{\gamma j}^\alpha \Gamma_{\beta i}^\gamma) \\ R_\beta^\alpha &= \sum_{i < j} R_{\beta ij}^\alpha dx^i \wedge dx^j. \end{cases} \quad (3.2.3)$$

The condition $R_{\beta ij}^\alpha = 0$ is the integrability condition of the system (3.1.5), in the classical sense of the word; it can be obtained by eliminating v^α from the equations given by substituting (3.1.5) into the identity $\partial_i \partial_j v^\alpha = \partial_j \partial_i v^\alpha$.

I.4 n^{th} -order differential equations

4.1. The solution of a linear homogeneous n^{th} -order differential equation

$$\frac{d^n}{dx^n} y = \sum_{i=1}^n a_i(x) \frac{d^{n-i}}{dx^i} y \quad (4.1.1)$$

is equivalent to that of the system

$$\begin{cases} \frac{d}{dx} y_i = y_{i+1} & (1 \leq i < n); \\ \frac{d}{dx} y_n = \sum_{i=1}^n a_i(x) y_{n+1-i} \end{cases} \quad (4.1.2)$$

of n first-order equations.

By §(I.3), this system can be described as the differential equation of horizontal sections of a rank- n vector bundle endowed with a suitable connection, and this is what we aim to further explore.

4.2. Let X be a non-singular complex-analytic variety of pure dimension 1. Let X_n be the n^{th} infinitesimal neighbourhood of the diagonal of $X \times X$, and p_1 and p_2 the two projections from X_n to X . We denote by $\pi_{k,l}$ the injection from X_l to X_k , for $l \leq k$.

Let $\Omega^{\otimes n}$ be the n^{th} tensor power of the invertible sheaf Ω_X^1 (for $n \in \mathbb{Z}$). If I is the ideal that defines the diagonal of $X \times X$, then $I/I^2 \simeq \Omega_X^1$ canonically, and

$$I^n/I^{n+1} \simeq \Omega^{\otimes n}. \quad (4.2.1)$$

If \mathcal{L} is an invertible sheaf on X , then we denote by $P^n(\mathcal{L})$ the vector bundle

$$P^n(\mathcal{L}) = (p_1)_* p_2^* \mathcal{L} \quad (4.2.2)$$

of n^{th} -order jets of sections of \mathcal{L} . The I -adic filtration of $p_2^* \mathcal{L}$ defines a filtration of $P^n(\mathcal{L})$ for which

$$\begin{aligned} \text{Gr} P^n(\mathcal{L}) &\simeq \text{Gr} P^n(\mathcal{O}) \otimes \mathcal{L} \\ \text{Gr}^i P^n(\mathcal{L}) &\simeq \Omega^{\otimes i} \otimes \mathcal{L} \quad (0 \leq i \leq n). \end{aligned} \quad (4.2.3)$$

Recall that we define, by induction on n , a *differential operator of order $\leq n$* as being a morphism $A: \mathcal{M} \rightarrow \mathcal{N}$ of abelian sheaves such that

$$\begin{cases} A \text{ is } \mathcal{O}\text{-linear} & \text{for } n = 0; \\ [A, f] \text{ is of order } \leq m \text{ for every local section } f \text{ of } \mathcal{O} & \text{for } n = m + 1. \end{cases}$$

For every local section s of \mathcal{L} , $p_2^* s$ defines a local section $D^n(s)$ of $P^n(\mathcal{L})$ (4.2.2). The \mathbb{C} -linear sheaf morphism $D^n: \mathcal{L} \rightarrow P^n(\mathcal{L})$ is the universal differential operator of order $\leq n$ with domain \mathcal{L} .

Definition 4.3. (i) A *linear homogeneous n^{th} -order differential equation* on \mathcal{L} is an \mathcal{O}_X -homomorphism $E: P^n(\mathcal{L}) \rightarrow \Omega^{\otimes n} \otimes \mathcal{L}$ that induces the identity on the submodule $\Omega^{\otimes n} \otimes \mathcal{L}$ of $P^n(\mathcal{L})$.

(ii) A local section s of \mathcal{L} is a *solution* of the differential equation E if $E(D^n(s)) = 0$.

In fact, I have cheated with this definition, in that I have only considered equations that can be put in the “resolved” form (4.1.1). | p. 25

4.4. Suppose that $\mathcal{L} = \mathcal{O}$, and let x be a local coordinate on X . The choice of x allows us to identify $P^k(\mathcal{O})$ with $\mathcal{O}^{[0,k]}$, with the arrow D^k becoming

$$D^k: \mathcal{O} \rightarrow P^k(\mathcal{O}) \simeq \mathcal{O}^{[0,k]}$$

$$f \mapsto (\partial_x^i f)_{0 \leq i \leq k}.$$

The choice of x also allows us to identify Ω^1 with \mathcal{O} , so that the n^{th} -order differential equation can be identified with a morphism $E \in \text{Hom}(\mathcal{O}^{[0,n]}, \mathcal{O})$, and, as such, has coordinates $(b_i)_{0 \leq i \leq n}$ with $b_n = 1$. The solutions of E are then exactly the (holomorphic) functions f that satisfy

$$\sum_{i=0}^n b_i(x) \partial_x^i f = 0 \quad (b_n = 1). \quad (4.4.1)$$

The existence and uniqueness theorem for solutions of the Cauchy problem in (4.4.1) implies the following.

Theorem 4.5. (Cauchy). *Let X and \mathcal{L} be as in (4.2), and let E be an n^{th} -order differential equation on \mathcal{L} . Then*

- (i) *the subsheaf of \mathcal{L} given by solutions of E is a local system \mathcal{L}^E of rank n on X ; and*
- (ii) *the canonical arrow $D^{n-1}: \mathcal{L}^E \rightarrow P^{n-1}(\mathcal{L})$ induces an isomorphism*

$$\mathcal{O} \otimes_{\mathbb{C}} \mathcal{L}^E \xrightarrow{\simeq} P^{n-1}(\mathcal{L}).$$

In particular, it follows from (4.5) (ii) and from (2.17) that E defines a canonical connection on $P^{n-1}(\mathcal{L})$, whose horizontal sections are the images under D of solutions of E .

4.6. To a differential equation E on \mathcal{L} , we have thus associated

- a) a holomorphic vector bundle \mathcal{V} endowed with a connection (which is automatically integrable) **!TODO!**; and
- b) a surjective homomorphism $\lambda: \mathcal{V} \rightarrow \mathcal{L}$ (by $i = 0$ in (4.2.3)).

Furthermore, the solutions of E are exactly the images under λ of the horizontal sections of \mathcal{V} . This is just another way of expressing how to obtain (4.1.2) from (4.1.1). | p. 26

4.7. Let \mathcal{V} be a rank- n vector bundle on X , endowed with a connection with covariant derivative ∇ . Let v be a local section of \mathcal{V} , and w a vector field on X that doesn't vanish at any point. We say that v is *cyclic* if the local sections $(\nabla_w)^i(v)$ of \mathcal{V} (for $0 \leq i < n$) form a basis of \mathcal{V} . This condition does not depend on the choice of w , and if f is an invertible holomorphic function, then v is cyclic if and only if fv is cyclic. In fact, we can show, by induction on i , that $(\nabla_{gw})^i(fv)$ lies in the submodule of \mathcal{V} generated by the $(\nabla_w)^j(v)$ (for $0 \leq j \leq i$).

If \mathcal{L} is an invertible module, then we say that a section v of $\mathcal{V} \otimes \mathcal{L}$ is cyclic if, for every local isomorphism between \mathcal{L} and \mathcal{O} , the corresponding section of \mathcal{V} is cyclic. This applies, in particular, to sections v of $\underline{\text{Hom}}(\mathcal{V}, \mathcal{L}) = \mathcal{V}^\vee \otimes \mathcal{L}$.

Lemma 4.8. *With the hypotheses and notation of (4.6), λ is a cyclic section of $\underline{\text{Hom}}(\mathcal{V}, \mathcal{L})$.*

Proof. The problem is local on X ; we can reduce to the case where $\mathcal{L} = \mathcal{O}$ and where there exists a local coordinate x .

We use the notation of (4.4). A section (f^i) of $P^{n-1}(\mathcal{O}) \simeq \mathcal{O}^{[0, n-1]}$ is horizontal if and only if it satisfies

$$\begin{cases} \partial_x f^i = f^{i+1} & (0 \leq i \leq n-2) \\ \partial_x f^{n-1} = -\sum_{i=0}^{n-1} b_i f^i. \end{cases}$$

This gives us the coefficients of the connection: the matrix of the connection is

$$\begin{pmatrix} 0 & -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & 0 & -1 & & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & 0 & -1 \\ b^0 & b^1 & \dots & \dots & b^{n-2} & b^{n-1} \end{pmatrix} \quad (4.8.1)$$

In the chosen system of coordinates, $\lambda = e^0$, and we calculate that

$$\nabla_x^i \lambda = e^i \quad (0 \leq i \leq n-1)$$

which proves (4.8). □

Proposition 4.9. *The construction in (4.6) establishes an equivalence between the following categories, where we take morphisms to be isomorphisms:*

- a) *the category of invertible sheaves on X endowed with an n^{th} -order differential equation (4.3); and*
- b) *the category of triples consisting of a rank- n vector bundle \mathcal{V} endowed with a connection, an invertible sheaf \mathcal{L} , and a cyclic homomorphism $\lambda: \mathcal{V} \rightarrow \mathcal{L}$.*

Proof. We will construct a functor that is quasi-inverse to that in (4.6). Let \mathcal{V} be a vector bundle with connection, and λ a homomorphism from \mathcal{V} to an invertible sheaf \mathcal{L} . We denote by V the local system of horizontal sections of \mathcal{V} . For every \mathcal{O} -module \mathcal{M} , we have (2.17)

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{V}, \mathcal{M}) \simeq \mathrm{Hom}_{\mathbb{C}}(V, \mathcal{M}).$$

In particular, we define a map γ^k from \mathcal{V} to $P^k(\mathcal{L})$ by setting, for any horizontal section v of \mathcal{V} ,

$$\gamma^k(v) = D^k(\lambda(v)). \quad (4.9.1)$$

Lemma 4.9.2. *The homomorphism λ is cyclic if and only if*

$$\gamma^{n-1}: \mathcal{V} \rightarrow P^{n-1}(\mathcal{L})$$

is an isomorphism.

The problem is local on X . We can restrict to the case where $\mathcal{L} = \mathcal{O}$ and where we have a local coordinate x . With the notation of (4.4), the morphism γ^k then admits the morphisms $\partial_x^i \lambda = \nabla_x^i$ (for $0 \leq i \leq k$) as coordinates. For $k = n - 1$, these form a basis of $\mathrm{Hom}(\mathcal{V}, \mathcal{O})$ if and only if γ^{n-1} is an isomorphism.

For $k \geq l$, the diagram

$$\begin{array}{ccc} & \mathcal{V} & \\ \gamma^k \swarrow & & \searrow v^l \\ P^k(\mathcal{L}) & \xrightarrow{\pi_{l,k}} & P^k(\mathcal{L}) \end{array} \quad (4.9.3)$$

commutes; if λ is cyclic, then this, along with (4.9.2), implies that $\gamma^n(v)$ is locally a direct factor, of codimension 1 in $P^n(\mathcal{L})$, and admits $\omega^{\otimes n} \otimes \mathcal{L} \simeq \mathrm{Ker}(\pi_{n-1,n})$ as a complement. There thus exists exactly one n^{th} -order differential equation

$$E: P^n(\mathcal{L}) \rightarrow \Omega^{\otimes n} \otimes \mathcal{L}$$

on \mathcal{L} such that $E \circ \gamma^n = 0$.

By (4.9.1), if v is a horizontal section of \mathcal{V} , then $ED^n \lambda v = E\gamma^n v = 0$, and so λv is a solution of E . We endow $P^{n-1}(\mathcal{L})$ with the connection (4.6) defined by E . If v is a horizontal section of \mathcal{V} , then $\gamma^{n-1}(v) = D^{n-1} \lambda v$, where λv is a solution of E , and $\gamma^{n-1}(v)$ is thus horizontal. We thus deduce that γ^{n-1} is compatible with the connections. A particular case of (4.9.3) shows that the diagram

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\gamma^{n-1}} & P^{n-1}(\mathcal{L}) \\ \lambda \searrow & & \swarrow \text{!TODO!} \\ & \mathcal{L} & \end{array}$$

commutes, whence we have an isomorphism between $(\mathcal{V}, \mathcal{L}, \lambda)$ and the triple given by (4.6) applied to (\mathcal{L}, E) . The functor

$$(\mathcal{V}, \mathcal{L}, \lambda) \mapsto (\mathcal{L}, E)$$

is thus quasi-inverse to the functor in (4.6). \square

4.10. We now summarise the relations between two systems $(\mathcal{V}, \mathcal{L}, \lambda)$ and (\mathcal{L}, E) that correspond under (4.6) and (4.9).

We have homomorphisms $\gamma^k: \mathcal{V} \rightarrow P^k(\mathcal{L})$, such that

$$(4.10.1) \quad \gamma^k(v) = D^k \lambda v \text{ for } v \text{ horizontal ;}$$

$$(4.10.2) \quad \gamma^0 = \lambda \text{ and } \pi_{l,k} \gamma^k = \gamma^l ;$$

$$(4.10.3) \quad \gamma^{n-1} \text{ is an isomorphism } (\lambda \text{ is cyclic}) ;$$

$$(4.10.4) \quad E\gamma^n = 0 ; \text{ and}$$

$$(4.10.5) \quad \lambda \text{ induces an isomorphism between the local system } V \text{ of sections of } \mathcal{V} \text{ and the local system } \mathcal{L}^E \text{ of solutions of } E.$$

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I.5 Second-order differential equations

In this section, we specialise the results of §(I.4) to the case where $n = 2$, and we express certain results given in R.C. Gunning [11] in a more geometric form.

5.1. Let S be an analytic space, and let $q: X_2 \rightarrow S$ be an analytic space over S that is locally isomorphic to the finite analytic space over S defined by the \mathcal{O}_S -algebra $\mathcal{O}_S[T]/(T^3)$.

The fact that the group PGL_2 acts **!TODO!** on \mathbb{P}^1 has the following infinitesimal analogue.

Lemma 5.2. *Under the hypotheses of (5.1), let u and v be S -immersions of X_2 into \mathbb{P}_S^1 , i.e.*

$$X_2 \begin{array}{c} \xrightarrow{u} \\ \xrightarrow{v} \end{array} \mathbb{P}_S^1.$$

*Then there exists exactly one **!TODO!** (S -automorphism) of \mathbb{P}_S^1 that sends u to v .*

Proof. The problem is local on S , which allows us to suppose that X_2 is defined by the \mathcal{O}_S -algebra $\mathcal{O}_S[T]/(T^3)$, and that $u(X_2)$ and $v(X_2)$ are contained inside the same affine line, say, \mathbb{A}_S^1 . By translation, we can assume that $u(0) = v(0) = 0$. We must then prove the existence and uniqueness of a **!TODO!** $p(x) = (ax + b)(cx + d)$ that satisfies $p(0) = 0$, with first derivative $p'(0) \neq 0$ and given second derivative $p''(0)$. We have that $b = 0$, and p can be written uniquely in the form

$$\begin{aligned} p(x) &= e \frac{x}{1 - fx} & (e \neq 0) \\ &= ex + efx^2 \pmod{x^3}. \end{aligned}$$

The claim then follows immediately. \square

5.3. By (5.2), there exists exactly (up to isomorphism) one pair (u, P) consisting of a projective line P on S (with structure group $\mathrm{PGL}_2(\mathcal{O}_S)$) and an S -immersion u of X_2 into P . We call P the *osculating projective line* of X_2 .

Let X be a smooth curve. Let X_2 the second infinitesimal neighbourhood of the diagonal of $X \times X$, and let q_1 and q_2 be the two projections from X_2 to X .

The morphism $q_1: X_2 \rightarrow X$ is of the type considered in (5.1).

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Definition 5.4. We define the *osculating projective line bundle of X* , denoted by P_{tg} , to be the osculating projective line bundle of $q_1: X_2 \rightarrow X$.

By definition, we thus have a canonical commutative diagram

$$\begin{array}{ccc} X_2 & \hookrightarrow & P_{\text{tg}} \\ & \searrow q_1 & \downarrow \\ & & X \end{array} \tag{5.4.1}$$

and, in particular, P_{tg} is endowed with a canonical section e , which is the image of the diagonal section of X_2 , and we have that

$$e^* \Omega_{P_{\text{tg}}/X}^1 \simeq \Omega_X^1. \tag{5.4.2}$$

5.5. If X is a projective line, then $\text{pr}_1: X \times X \rightarrow X$ is a projective bundle on X such that P_{tg} can be identified with the constant projective bundle of fibre X on X endowed with the inclusion homomorphism of X_2 into $X \times X$, i.e.

$$\begin{array}{ccc} X_2 & \hookrightarrow & X \times X \\ & \searrow q_1 & \downarrow \text{pr}_1 \\ & & X \end{array}$$

In this particular case, we have a canonical commutative diagram

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$$\begin{array}{ccc} X_3 & \hookrightarrow & P_{\text{tg}} \\ \uparrow & \searrow & \downarrow \\ X_2 & \hookrightarrow & X \end{array}$$

Now let X be an arbitrary smooth curve.

Definition 5.6. (Local version). A *projective connection* on X is a sheaf on X of germs of local isomorphisms from X to \mathbb{P}^1 , which is a principal homogeneous sheaf (i.e. a torsor) for the constant sheaf of groups with value $\text{PLG}_2(\mathbb{C})$.

If X is endowed with a projective connection, then every local construction on \mathbb{P}^1 that is invariant under the projective group can be transported to X ; in particular, the construction in (5.5) gives us a morphism γ that fits into a commutative diagram:

$$\begin{array}{ccc} X_3 & \xrightarrow{\gamma} & P_{\text{tg}} \\ \uparrow & \searrow & \downarrow \\ X_2 & \hookrightarrow & X \end{array} \tag{5.6.1}$$

It is not difficult to show that such a morphism γ is defined by a unique projective connection (a proof of this will be given in (5.10)), and so Definition (5.6) is equivalent to the following.

Definition 5.6 bis. (*Infinitesimal version*). A projective connection on X is a morphism $\gamma: X_3 \hookrightarrow P_{\text{tg}}$ that makes the diagram in (5.6.1) commute.

Intuitively, giving a projective connection (the infinitesimal version) allows us to define **!TODO!**

5.7. Set $\Omega^{\otimes n} = (\Omega_X^1)^{\otimes n}$ (4.2). The sheaf of ideals on X_3 that defines X_2 is canonically isomorphic to $\Omega^{\otimes 3}$, and is annihilated by the sheaf of ideals that defines the diagonal. Also, if Δ is the diagonal map, then, by (5.4.2), we have that | p. 32

$$\Delta^* \gamma^* \Omega_{P_{\text{tg}}/X}^1 \simeq \Omega^1.$$

We thus deduce that the set of X -homomorphisms from X_3 to P_{tg} that induce the canonical homomorphism from X_2 to P_{tg} is either empty, or a principal homogeneous space for

$$\text{Hom}_X(\Delta^* \gamma^* \Omega_{P_{\text{tg}}/X}^1, \Omega^{\otimes 3}) = \text{Hom}_X(\Omega^1, \Omega^{\otimes 3}) = H^0(X, \Omega^{\otimes 2}).$$

If we replace X by a small enough open subset, then this set is non-empty:

Proposition 5.8. *Projective connections of open subsets of X form a principal homogeneous sheaf (i.e. a torsor) for the sheaf $\Omega^{\otimes 2}$.*

Proof. If η is a section of $\Omega^{\otimes 2}$, and $\gamma_1: X_3 \rightarrow P_{\text{tg}}$ is a projective connection, then the connection $\gamma_2 = \gamma_1 + \eta$ is defined, for any function f on P_{tg} , by

$$\gamma_2^* f = \gamma_1^* f + \eta \cdot e^* df \quad (5.8.1)$$

(modulo the identification of $\Omega^{\otimes 3}$ with an ideal of \mathcal{O}_{X_3}). □

5.9. Let $f: X \rightarrow Y$ be a homomorphism between smooth curves endowed with projective connections γ_X and γ_Y , and suppose that f is a local isomorphism (i.e. $df \neq 0$ at all points). Set

$$\theta f = f^* \gamma_Y - \gamma_X \in \Gamma(X, \Omega^{\otimes 2}).$$

For a composite map $g \circ f: X \xrightarrow{f} Y \xrightarrow{g} Z$, we trivially have that

$$\begin{aligned} \theta(g \circ f) &= \theta(f) + f^* \theta(g), \\ \text{whence } \theta(f^{-1}) &= -f^* \theta(f). \end{aligned} \quad (5.9.1)$$

Suppose that X and Y are open subsets of \mathbb{C} , and endowed with the projective connection induced by that of $\mathbb{P}^1(\mathbb{C})$. Denoting by x the injection from X into \mathbb{C} , we then have that | p. 33

$$\theta f = \frac{f'(f'''/6) - (f''/2)^2}{(f')^2} dx^{\otimes 2}. \quad (5.9.2)$$

To prove this, we identify, using (5.5), the projective double tangent bundle to X or Y with the constant projective bundle. The morphism $\delta f: P_{\text{tg},X} \rightarrow P_{\text{tg},Y}$ induced by f can be written as

$$\delta f: (x, p) \mapsto \left(f(x), f(x) + \frac{f'(x)(p-x)}{1 - \frac{1}{2}(f''(x)/f'(x))(p-x)} \right).$$

Consider the diagram

$$\begin{array}{ccc} X_3 & \xrightarrow{f} & Y_3 \\ \downarrow & & \downarrow \\ P_{\text{tg},X} & \xrightarrow{\delta f} & P_{\text{tg},Y} \end{array}$$

Then $\theta(f)$ describes the lack of commutativity of the diagram, i.e. the difference between the jets

$$(x, x + \varepsilon) \mapsto \left(f(x), f(x) + f'(x)\varepsilon + f''(x)\frac{\varepsilon^2}{2} + f'''(x)\frac{\varepsilon^3}{6} \right) \quad (\varepsilon^4 = 0)$$

and

$$\begin{aligned} (x, x + \varepsilon) &\mapsto \left(f(x), f(x) + \frac{f'(x)\varepsilon}{1 - \frac{1}{2}(f''(x)/f'(x))\varepsilon} \right) \\ &= \left(f(x), f(x) + f'(x)\varepsilon + f''(x)\frac{\varepsilon^2}{2} + \frac{1}{4}(f''(x)^2/f'(x))\varepsilon^3 \right). \end{aligned}$$

We thus have that

$$\theta(f) = \left(\frac{1}{6}f'''(x) - \frac{1}{4}f''(x)^2/f'(x) \right) dx^{\otimes 3} df^{\otimes -1},$$

and (5.9.2) then follows.

Equation (5.9.2) shows that $6\theta f$ is the classical *Schwarz derivative* of f .

If a map f from $X \subset \mathbb{C}$ to $\mathbb{P}^1(\mathbb{C})$ is described by projective coordinates $f = (g, h)$, then

$$\theta(f) = \frac{\begin{vmatrix} g & g' \\ h & h' \end{vmatrix} \left(\begin{vmatrix} g & g'''/6 \\ h & h'''/6 \end{vmatrix} + \begin{vmatrix} g' & g'/2 \\ h' & h'/2 \end{vmatrix} \right) - \begin{vmatrix} g & g''/2 \\ h & h''/2 \end{vmatrix}^2}{\begin{vmatrix} g & g' \\ h & h' \end{vmatrix}^2} \quad (5.9.3)$$

To prove (5.9.3), the simplest method is to note the following.

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- (i) The right-hand side of (5.9.3) is invariant under a linear substitution of constant coefficients L acting on g and h : the numerator and denominator $\det(L)^2$.
- (ii) The right-hand side of (5.9.3) is invariant under the substitution

$$(g, h) \mapsto (\lambda g, \lambda h).$$

Denoting the determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by $\det(a, b)$, we have that

$$\begin{aligned} \det(g, (\lambda g)') &= \det(\lambda g, \lambda g' + \lambda' g) \\ &= \lambda^2 \det(g, g') \\ \det(\lambda g, (\lambda g)''/2) &= \lambda^2 \det(g, g''/2) + \lambda \lambda' \det(g, g') \\ \det(\lambda g, (\lambda g)'''/6) &= \lambda^2 \det(g, g'''/6) + \lambda \lambda' \det(g, g''/2) \\ &\quad + (\lambda \lambda''/2) \det(g, g') \\ \det((\lambda g)', (\lambda g)''/2) &= \lambda^2 \det(g', g''/2) - (\lambda \lambda''/2) \det(g, g') \\ &\quad + (\lambda')^2 \det(g, g') + \lambda \lambda' \det(g, g''/2). \end{aligned}$$

The new denominator D_λ (resp. numerator N_λ) is thus given in terms of the old denominator D (resp. numerator N) by

$$D_\lambda = \lambda^4 D$$

$$N_\lambda = \lambda^4 N + \lambda^4 \det(g, g') \cdot \begin{bmatrix} \left(\lambda'/\lambda \det(g, g''/2) + (\lambda''/2\lambda) \det(g, g') \right) \\ + \left(-(\lambda''/2\lambda) \det(g, g') + (\lambda'/\lambda)^2 \det(g, g') + (\lambda'/\lambda) \det(g, g'') \right) \\ - \left(2(\lambda'/\lambda) \det(g, g''/2) + (\lambda'/\lambda)^2 \det(g, g') \right) \end{bmatrix}$$

and $N_\lambda/D_\lambda = N/D$.

With these variance properties agreeing with those of the left-hand side of (5.9.3), it suffices to prove (5.9.3) in the particular case where $h = 1$. The equation then reduces to (5.9.2).

We will only need to use the fact that $\theta(f)$ can be expressed in terms of **!TODO!** : we have, for $Z_i \sim Z$,

$$\frac{(f(Z_1), f(Z_2), f(Z_3), f(Z_4))}{(Z_1, Z_2, Z_3, Z_4)} - 1 = \theta(f)(Z_1 - Z_2)(Z_3 - Z_4) + \mathcal{O}((Z_i - Z)^3). \quad (5.9.4)$$

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5.10. The differential equation $\theta(f) = 0$ (for $f: X \rightarrow \mathbb{P}^1(\mathbb{C})$ with non-zero first derivative) is a third-order differential equation. It thus admits ∞^3 solutions, locally, and these solutions are permuted amongst themselves by the projective group (since the group transitively permutes the Cauchy data: (5.2)). The set of solutions is thus a projective connection (the local version, (5.6)). This construction is inverse to that which associates, to any projective connection in the sense of (5.6), a projective connection in the sense of (5.6 bis).

5.11. Let X be a smooth curve, \mathcal{L} an invertible sheaf on X , and E a second-order ordinary differential equation on \mathcal{L} . We have seen, in (4.5), that E defines a connection on the bundle $P^1(\mathcal{L})$ of first-order jets of sections of \mathcal{L} , and we obtain from it a connection on

$$\bigwedge^2 P^1(\mathcal{L}) \simeq \mathcal{L} \otimes \Omega^1(\mathcal{L}) = \Omega^1 \otimes \mathcal{L}^{\otimes 2}.$$

If X is a compact connected curve of genus g , then the bundle $\Omega^1 \otimes \mathcal{L}^{\otimes 2}$ is thus necessarily of degree 0, and we have that

$$\deg(\mathcal{L}) = 1 - g.$$

Let V be the rank-2 local system of solutions of E ; we have (4.5) that $\mathcal{O} \otimes V \simeq P^1(\mathcal{L})$, and the linear form $\lambda: P^1(\mathcal{L}) \rightarrow \mathcal{L}$ defines a section λ_0 of the projective bundle associated to the vector bundle $P^1(\mathcal{L})$.

Locally on X , V is isomorphic to the constant local system \mathbb{C}^2 ; the choice of an isomorphism $\sigma: V \rightarrow \mathbb{C}^2$ identifies λ_0 with a map $\lambda_{0,\sigma}$ from X to $\mathbb{P}^1(\mathbb{C})$; by (4.8), the differential of this map is everywhere non-zero, and so $\lambda_{0,\sigma}$ allows us to transport the canonical projective connection of $\mathbb{P}^1(\mathbb{C})$ to X . This connection does not depend on the choice of σ , and so the differential equation E defines a projective connection on X .

Proposition 5.12. *Let \mathcal{L} be an invertible sheaf on a smooth curve X . The construction in (5.10) gives a bijection between*

- a) *the set of second-order ordinary differential equations on \mathcal{L} ; and*
- b) *the set of pairs consisting of a projective connection on X and a connection on $\Omega^1(\mathcal{L}^{\otimes 2})$.*

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Proof. The problem is local on X ; we can thus suppose that X is an open subset of \mathbb{C} , and that $\mathcal{L} = \mathcal{O}$. An equation E can then be written as

$$E: y'' + a(x)y' + b(x)y = 0.$$

If we identify $P^1(\mathcal{L})$ with \mathcal{O}^2 , then the matrix of the connection (5.10) defined by E on $\wedge^2 P^1(\mathcal{L}) \sim \mathcal{O}$ is then $-a(x)\text{tr}M$, where M is the matrix in (4.8.1).

Let φ be the identity map from X , an open subset of $\mathbb{P}^1(\mathbb{C})$, to itself endowed with the projective connection defined by E . Identifying Ω with \mathcal{O} by means of the given local coordinate, we then have that

$$\theta(\varphi) = \frac{1}{3}b - \frac{1}{12}(a^2 + 2a'). \quad (5.12.1)$$

Indeed, if f and g are two linearly independent solutions of E , then the map with projective coordinates

$$(f, g): X \rightarrow \mathbb{P}^1(\mathbb{C})$$

!TODO! the projective connection. We have that

$$\begin{aligned} f'' &= -(af' + bf) \\ f''' &= -a(-af' - bf) - bf' - a'f' - b'f \\ &= (a^2 - a' - b)f' + (ab - b')f. \end{aligned}$$

Equation (5.9.3) gives (using the same notation for determinants as before)

$$\begin{aligned} \theta(\varphi) &= \frac{\det(f, f') \left(\frac{1}{6}(a^2 - a' - b)\det(f, f') + \frac{1}{2}b\det(f, f') \right) - \left(\frac{1}{2}a \right)^2 \det(f, f')^2}{\det(f, f')^2} \\ &= \frac{1}{6}(a^2 - a' - b) + \frac{1}{2}b - \frac{1}{4}a^2 \\ &= \frac{1}{3}b - \frac{1}{12}(a^2 + 2a'). \end{aligned}$$

We conclude by noting that (a, b) is uniquely determined by $(-a, \frac{1}{3}b - \frac{1}{12}(a^2 + 2a'))$, and that, for any holomorphic function g on an open subset U of \mathbb{C} , there exists a unique projective connection on U satisfying $\theta(\varphi) = g$, for φ **!TODO!** the connection (same proof as for (5.10), or (5.8)). \square

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I.6 Multiform functions of finite determination

6.1. Let X be a non-empty connected topological space that is both locally path connected and locally simply path connected, and let x_0 be a point of X . We denote by $\pi: \tilde{X}_{x_0} \rightarrow X$ the universal covering of (X, x_0) , and by \tilde{x}_0 the base point of \tilde{X}_{x_0} .

If \mathcal{F} is a sheaf on X , then we pose:

Definition 6.2. A *multiform section* of \mathcal{F} on X is a global section of the inverse image $\pi^*\mathcal{F}$ of \mathcal{F} on \tilde{X}_{x_0} .

If s is a multiform section of \mathcal{F} on X , then a *determination of s at a point x of X* is an element of the fibre $\mathcal{F}_{(x)}$ of \mathcal{F} at x that is an inverse image of s under a local section of π at x . **!TODO! (check)** Each point in $\pi^{-1}(x)$ thus defines a determination of s at x . We define the *base determination* of s at x_0 to be the determination defined by \tilde{x}_0 . We define a *determination of s on an open subset U of X* to be a section of \mathcal{F} over U whose **!TODO!** at every point of U is a determination of s at that point.

Definition 6.3. We say that \mathcal{F} satisfies the *principle of analytic continuation* if the set where any two local sections of \mathcal{F} agree is always (open and) closed.

Example 6.4. If \mathcal{F} is a coherent analytic sheaf on a complex-analytic space, then \mathcal{F} satisfies the principle of analytic continuation if and only if \mathcal{F} has no **!TODO!** components.

Proposition 6.5. Let X and x_0 be as in (5.1), and let \mathcal{F} be a sheaf of \mathbb{C} -vector spaces on X that satisfies the principle of analytic continuation. For every multiform section s of \mathcal{F} , the following conditions are equivalent:

- (i) the determinations of s at x_0 generate a finite-dimensional sub-vector space of \mathcal{F}_{x_0} ; and
- (ii) the subsheaf of \mathcal{F} of \mathbb{C} -vector spaces generated by the determinations of s is a complex local system (1.1).

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Proof. It is trivial that (ii) implies (i). We now prove that (i) implies (ii). Let x be a point of X at which the determinations of s generate a finite-dimensional sub-vector space of \mathcal{F}_x , and let U be a connected open neighbourhood of x over which \tilde{X}_{x_0} is trivial: $(\pi^{-1}(U), \pi) \simeq (U \times I, \text{pr}_1)$ for some suitable set I . We will prove that, over U , the determinations of s generate a complex local system. Each $i \in I$ defines a determination s_i of s , and, over U , the vector subsheaf of \mathcal{F} generated by the determinations of s is generated by the $(s_i)_{i \in I}$; if this sheaf is constant, then the hypotheses on x implies that it is a local system. We have:

Lemma 6.6. *If a sheaf \mathcal{F} of \mathbb{C} -vector spaces on a connected space satisfies the principle of analytic continuation, then the vector subsheaf of \mathcal{F} generated by a family of global sections s_i is a constant sheaf.*

The sections s_i define

$$a: \underline{\mathbb{C}}^{(I)} \rightarrow \mathcal{F}$$

with the image being the vector subsheaf \mathcal{G} of \mathcal{F} generated by the s_i . If an equation $\sum_i \lambda_i s_i = 0$ between the s_i holds at a point, then it holds everywhere, by the principle of analytic continuation.

The sheaf $\text{Ker}(a)$ is thus constant subsheaf of $\underline{\mathbb{C}}^{(I)}$, and the claim then follows.

We conclude the proof of (6.5) by noting that, by the above, the largest open subset of X over which the determinations of s generate a local system is closed and contains x_0 . \square

Definition 6.7. Under the hypotheses of (6.5), a multiform section s of \mathcal{F} is said to be a *finite determination* if it satisfies either of the equivalent conditions of (6.5).

6.8. Under the hypotheses of (6.5), let s be a multiform section of finite determination of \mathcal{F} . This section defines

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- a) the local system V generated by its determinations ;
- b) a **!TODO!** of V at x_0 , say, v_0 , corresponding to the base determination of s ; and
- c) an inclusion morphism $\lambda: V \rightarrow \mathcal{F}$.

The triple consisting of V_{x_0} , v_0 , and the representation of $\pi_1(X, x_0)$ on V_{x_0} defined by V (1.4) is called the *monodromy* of s . The triple (V, V_0, λ) satisfies the following two conditions.

(6.8.1) v_0 is a cyclic vector of the $\pi_1(X, x_0)$ -module V_{x_0} , i.e. it generates the $\pi_1(X, x_0)$ -module V_{x_0} .

This simply means that V is generated by the set of determinations of the unique multiform section of V with base determination v_0 .

(6.8.2) $\lambda: V_{x_0} \rightarrow \mathcal{F}_{x_0}$ is injective.

6.9. Let W_0 be a finite-dimensional complex representation of $\pi_1(X, x_0)$, endowed with a cyclic vector w_0 . The multiform section s of \mathcal{F} is said to be *monodromy subordinate to (w_0, v_0)* if it is the finite determination, and if, with the notation of (6.8), there exists a homomorphism of $\pi_1(X, x_0)$ -representations of W_0 in V_{x_0} that sends w_0 to v_0 . Let W be the local system defined by W_0 , and let w be the unique multiform section of w of base determination w_0 . It is clear that, under the hypotheses of (6.5), we have

Proposition 6.10. *The function $\lambda \mapsto \lambda(w)$ is a bijection between the set $\text{Hom}_{\mathbb{C}}(W, \mathcal{F})$ and the set of multiform sections of \mathcal{F} with monodromy subordinate to (W_0, w_0) .*

Corollary 6.11. *Let X be a reduced connected complex analytic space endowed with a base point x_0 . Let W_0 be a finite-dimensional complex representation of $\pi_1(X, x_0)$ endowed with a cyclic vector w_0 , and W the corresponding local system on X , with $\mathcal{W} = \mathcal{O} \otimes_{\mathbb{C}} W$ being the*

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associated vector bundle, and w the unique multiform section of \mathcal{W} of base determination w_0 . Write \mathcal{W}^\vee be the dual vector bundle of \mathcal{W} . Then the function

$$\lambda \mapsto \langle \lambda, w \rangle,$$

from $\Gamma(X, \mathcal{W}^\vee)$ to the set of multiform holomorphic functions on X of monodromy subordinate to (W_0, w_0) , is a bijection.

Corollary 6.12. *If X is Stein, then there exist multiform holomorphic functions on X of any given monodromy (W_0, w_0) .*

Chapter II

Regular connections

II.1 Regularity in dimension 1

1.1. Let U be an open neighbourhood of 0 in \mathbb{C} , and consider an n^{th} -order differential equation

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$$y^{(n)} + \sum_{i=0}^{n-1} a_i(x)y^{(i)} = 0 \quad (1.1.1)$$

where the a_i are holomorphic functions on $U \setminus \{0\}$. We classically say that 0 is a *regular singular point* of (1.1.1) if the functions $x^{n-i}a_i(x)$ are holomorphic at 0. If this is true, then, after multiplying by x^n , we can write (1.1.1) in the form

$$\left(x \frac{d}{dx}\right)^n y + \sum b_i(x) \left(x \frac{d}{dx}\right)^i y = 0 \quad (1.1.2)$$

where the $b_i(x)$ are holomorphic at 0.

In this section, we will translate this idea into the language of connections (cf. §(I.4)), and we will establish some of its properties.

The results in this section were taught to me by N. Katz. They are either due to N. Katz (see, most notably, [14, 15]), or classical (see, for example, Ince [13], and Turrittin [24, 25]).

1.2. Let K be a (commutative) field, Ω a rank-1 vector space over K , and $d: K \rightarrow \Omega$ a non-trivial derivation, i.e. an non-zero additive map that satisfies the identity

$$d(xy) = xdy + ydx. \quad (1.2.1)$$

Let V be an n -dimensional vector space over K . Then a *connection* on V is an additive map $\nabla: V \rightarrow \Omega \otimes V$ that satisfies the identity

$$\nabla(xv) = dx \cdot v + x\nabla v. \quad (1.2.2)$$

If τ is an element of the dual Ω^\vee of Ω , then we set

$$\partial_\tau(x) = \langle dx, \tau \rangle \in K, \quad (1.2.3)$$

$$\nabla_\tau(v) = \langle dv, \tau \rangle \in V. \quad (1.2.4)$$

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We thus have that

(1.2.5) ∂_τ is a derivation;

(1.2.6) $\nabla_\tau(xv) = \partial_\tau(x) \cdot v + x\nabla_\tau v$; and

(1.2.7) $\nabla_{\lambda\tau}(v) = \lambda\nabla_\tau v$.

Let $v \in V$. We can easily show that the vector subspace of V generated by the vectors

$$v, \nabla_{\tau_1} v, \nabla_{\tau_2} \nabla_{\tau_1} v, \dots, \nabla_{\tau_k} \cdots \nabla_{\tau_1} v$$

(where $\tau_i \neq 0$ in Ω) does not depend on the choice of the $\tau_i \neq 0$, and does not change if we replace v by λv (for some $\lambda \in K^*$). Furthermore, if the last of these vectors is a linear combination of the preceding vectors, then this vector subspace is stable under derivations. We say that v is a *cyclic vector* if, for $\tau \in \Omega$, the vectors

$$\nabla_\tau^i v \quad (0 \leq i \leq n)$$

form a basis of V .

Lemma 1.3. *Under the above hypotheses, and if K is of characteristic 0, then there exists a cyclic vector.*

Proof. Let $t \in K$ be such that $dt \neq 0$, and let $\tau = t/dt \in \Omega^\vee$. Then $\partial_\tau(t^k) = kt^k$.

Let $m \leq n$ be the largest integer such that there exists a vector e such that the vectors $\partial_\tau^i e$ (for $0 \leq i \leq m$) are linearly independent. If $m \neq n$, then there exists a vector f that is linearly independent of the $\partial_\tau^i e$. For any rational number λ and integer k , the vectors

$$\partial_\tau^i(e + \lambda t^k f) \quad (0 \leq i \leq m)$$

are linearly dependent, and their exterior product $\omega(\lambda, k)$ is thus zero. We have that

$$\partial_\tau^i(e + \lambda t^k f) = \partial_\tau^i e + \sum_{0 \leq k \leq i} k^j t^k \partial_\tau^{i-j} f.$$

From this equation, we obtain a finite decomposition

$$\omega(\lambda, k) = \sum_{\substack{0 \leq a \leq m \\ 0 \leq b}} \lambda^a t^{ka} k^b \omega_{a,b}$$

where $\omega_{a,b}$ is independent of λ and k . Since $\omega(\lambda, k) = 0$ for all $\lambda \in \mathbb{Q}$, and since

$$\omega(\lambda, k) = \sum \lambda^a \omega_a(k)$$

where $\omega_a(k) = t^{ka} (\sum k^b \omega_{a,b}) = t^{ka} \omega'_a(k)$, we have that $\omega_a(k) = \omega'_a(k) = 0$. Since

$$\omega'_a(k) = \sum k^b \omega_{a,b} = 0$$

for all $k \in \mathbb{Z}$, we have that $\omega_{a,b} = 0$. In particular,

$$\omega_{1,m} = e \wedge \partial_\tau^1 e \wedge \dots \wedge \partial_\tau^{m-1} e \wedge f = 0,$$

and f is then linearly dependent of the $\partial_\tau^i e$ ($0 \leq i \leq m$), which contradicts the hypothesis. Thus $m = n$, and so e is a cyclic vector \square

1.4. Let \mathcal{O} be a discrete valuation ring of *equal characteristic* 0, with maximal ideal \mathfrak{m} , residue field $k = \mathcal{O}/\mathfrak{m}$, and field of fractions K . Suppose that \mathcal{O} is endowed with a free rank-1 \mathcal{O} -module Ω along with a derivation $d: \mathcal{O} \rightarrow \Omega$ that satisfies

1.4.1. *There exists a uniformiser t such that dt generates Ω .*

(For less hyper-generality, see ??).

If t_1 is another uniformiser, then $t_1 = at$ for some $a \in \mathcal{O}^*$, and, by hypothesis, da is a multiple of dt , i.e. $da = \lambda dt$. We thus have that

$$dt_1 = adt + da \cdot t = (a + \lambda t)dt$$

and so

1.4.2. *For every uniformiser t , dt generates Ω .*

We denote by

$$v: K^* \rightarrow \mathbb{Z}$$

the valuation of K defined by \mathcal{O} ; we also denote by v the valuation of $\Omega \otimes K$ defined by the lattice Ω . If t is a uniformiser, then

$$v(\omega) = v(\omega/dt).$$

If $f \in K^*$ with $f = at^n$ (for $a \in \mathcal{O}$), then

$$df = da \cdot t^n + nat^{n-1}dt$$

and thus

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$$(1.4.3) \quad v(df) \leq v(f) - 1; \text{ and}$$

$$(1.4.4) \quad v(f) \neq 0 \implies v(df) = v(f) - 1.$$

In particular, d is continuous and extends to $d: \mathcal{O}^\wedge \rightarrow \Omega^\wedge$, and the triple $(\mathcal{O}^\wedge, d, \Omega^\wedge)$ again satisfies (1.4.1).

Lemma 1.5. *If \mathcal{O} is complete, then the triple (\mathcal{O}, d, Ω) is isomorphic to the triple $(k[[t]], \partial_t, k[[t]])$.*

Proof. The homomorphisms

$$\text{Gr}(d): \mathfrak{m}^i/\mathfrak{m}^{i+1} \rightarrow \mathfrak{m}^{i-1}\Omega/\mathfrak{m}^i\Omega$$

induced by d are linear and bijective (1.4.4). Since \mathcal{O} is complete, $d: \mathfrak{m} \rightarrow \Omega$ is surjective, and $\text{Ker}(d) \simeq k$. This gives us a field of representatives that is annihilated by d , and the choice of a uniformiser t gives the desired isomorphism $k[[t]] \simeq \mathcal{O}$. \square

1.6. If an \mathcal{O} -algebra \mathcal{O}' is a discrete valuation ring with a field of fractions K' that is algebraic over K , then the derivation d extends uniquely to $d: K' \rightarrow \Omega \otimes_{\mathcal{O}} K'$. Let e be the ramification index of \mathcal{O}' over \mathcal{O} , and let t' be a uniformiser of \mathcal{O}' . We set

$$\Omega' = 1/(t')^{e-1}\Omega \otimes_{\mathcal{O}} \mathcal{O}'.$$

We can easily show, using (1.6), that the triple $(\mathcal{O}', d, \Omega')$ again satisfies (1.4.1).

1.7. We will mostly be interested in the following examples. Let X be a non-singular complex algebraic curve, and let $x \in X$. We choose one of the following:

(1.7.1) $\mathcal{O} = \mathcal{O}_{x,X}$ (the local ring for the Zariski topology), $\Omega = (\Omega_{X/\mathbb{C}}^1)_x$, and d = the differential ;

(1.7.2) $\mathcal{O} = \mathcal{O}_{x,X^{\text{an}}}$ (the local ring of germs at x of holomorphic functions), $\Omega = (\Omega_{X^{\text{an}}/\mathbb{C}}^1)_{(x)}$, and d = the differential ; or

(1.7.3) the common completion of (1.7.1) and (1.7.2).

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1.8. Under the hypotheses of (1.4), let V be a finite-dimensional vector space over K , and V_0 a lattice in V , i.e. a free sub- \mathcal{O} -module of V such that $KV_0 = V$. For every homomorphism $e: \mathcal{O}^n \rightarrow V$, we define the *valuation* $v(e)$ of e to be the largest integer m such that $e(\mathcal{O}^n) \subset m^m V_0$. If V_0 and V_1 are two lattices, then there exists an integer s that is independent of e and n and is such that

$$|v_0(e) - v_1(e)| \leq s. \quad (1.8.1)$$

Theorem 1.9. [N. Katz]. *Under the hypotheses of (1.4), and with the notation of (1.8), let ∇ be a connection (1.2) on a vector space V of dimension n over K . Then one of the following conditions is satisfied:*

- a) *For any lattice V_0 in V , any basis $e: K^n \xrightarrow{\sim} V$ of V , any differential form with a simple pole ω and $\tau = \omega^{-1} \in \Omega_K$, the numbers $-v(\nabla_\tau^i e)$ are bounded above ; or*
- b) *There exists a rational number $r > 0$, with denominator at most n , such that, for any V_0 , e , and τ as above, the family of numbers*

$$|-v(\nabla_\tau^i e) - ri|$$

is bounded.

Conditions a) and b) of (1.9) are more manageable in a different form:

Lemma 1.9.1. *Let V_0 , τ , and e be as in (1.9). Then, for any given value of r , the estimation b) (!TODO! ??) is equivalent to*

$$|\sup_{j \leq i} (-v \nabla_\tau^j e) - ri| \leq C^{te}. \quad (1.9.2)$$

The estimation a) (!TODO! ??) is equivalent to the same bound (1.9.2) for $r = 0$.

Proof. Going from (1.9) to (1.9.2) is clear, as is the converse for $r = 0$. So suppose that (1.9.2) holds true for $r > 0$ and some value C_0 for the constant. We have

$$-v \nabla_\tau^i e - ri \leq C_0. \quad (a)$$

We immediately see that there exists a constant k such that

$$-v \nabla_\tau^n (\nabla_\tau^i e) \leq -v \nabla_\tau^i e + kn.$$

Thus

$$\begin{aligned}
-C_0 + r(i+n) &\leq \sup_{j \leq i+n} -v \nabla_{\tau}^j e \\
&= \sup_{j \leq i} (\sup_{j \leq i} -v \nabla_{\tau}^j e, -v \nabla_{\tau}^i e + kn) \\
&\leq \sup(C_0 + ri, -v \nabla_{\tau}^i e + kn)
\end{aligned}$$

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and, if $-C_0 + r(i+n) > C_0 + ri$, i.e. if $n > 2C_0/r$, then we have

$$-v \nabla_{\tau}^i e \geq (-C_0 - kn - rn) + ri. \quad (\text{b})$$

The inequalities (a) and (b) then imply the inequality of the form (1.9):

$$|-v \nabla_{\tau}^i e - ri| \leq C_0 + kn + rn.$$

□

Lemma 1.9.3. *Let (V_0, τ_0, e_0) and (V_1, τ_1, e_1) be two systems as in (1.9). Then*

$$|\sup_{j \leq i} (-v_1 \nabla_{\tau_1}^j e_1) - \sup_{j \leq i} (-v_0 \nabla_{\tau_0}^j e_0)| \leq C^{te}.$$

Proof.

□

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