Multialgebraic categories

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Translator's note.

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Contents

0 Introduction

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We revise the notions of algebraic theories, algebras, categories, and algebraic functors introduced by F.W. Lawvere, in such a way that the essential theorems can be generalised to apply to non-algebraic situations, such as that of fields, local rings, totally ordered sets, metric spaces, normed vector spaces, pre-Hilbert spaces, etc.

A multialgebraic category is a category of functors that are multicontinuous for finite multiproducts, defined over a small category with finite multiproducts with values in $\mathbb{E}\mathrm{ns}$.

[*] https://thosgood.com/translations

We show that multialgebraic categories have filtered colimits, connected limits, and cokernels for coequalisable pairs of morphisms, and that their equivalence relations are effective, and their regular epimorphisms are universal, and that they have regular universal factorisations. They are equipped with a structure-forgetful functor with values in $\mathbb{E}$ns, which admits a left multiadjoint, reflects isomorphisms, and preserves filtered colimits, connected limits, and regular epimorphisms. We give two characterisations of multialgebraic categories, and we show that they are equivalent to multimonadic categories of finite rank over $\mathbb{E}$ns. Proper morphisms of multialgebraic theories determine proper multialgebraic functors. These functors possess a left adjoint. For example, the inclusion functors of the category of commutative fields into the category of integral domains and into the category of commutative local rings are both proper multialgebraic.

We use the notation and results of [2] and [3]. *p. 194*

1 Multialgebraic theories and multialgebras

1.0 Definition [2]. A *multiproduct* of a small family $(X_i)_{i\in I}$ of objects of a category $\mathbb{A}$ is a small family $(\gamma_{ij}\colon Y_j \to X_i)_{(i,j)\in I\times J}$ of morphisms in $\mathbb{A}$ such that, for every family $(f_i\colon Y \to X_i)_{i\in I}$ of morphisms in $\mathbb{A}$, there exists a unique pair (j,f) consisting of $j\in J$ and a morphism $f\colon Y\to Y_j$ such that $\gamma_{ij}f = f_i$ for all $i\in I$.

We say that the Y_j *belong* to the multiproduct of objects $(X_i)_{i\in I}$. The multiproduct is said to be *finite* if I is finite. The category $\mathbb{A}$ is said to have *finite multiproducts* if every finite family of objects of $\mathbb{A}$ has a multiproduct.

1.1 Definitions. A *multialgebraic theory* is a small category $\mathbb{M}$ with finite multiproducts, endowed with a distinguished small family of objects $(X_g)_{g\in G}$ such that every object of $\mathbb{M}$ belongs to a finite multiproduct of objects of this family.

An $\mathbb{M}$-*multialgebra* is a functor $F\colon \mathbb{M}\to\mathbb{E}\mathrm{ns}$ that is multicontinuous for finite multiproducts [2], i.e. for every finite sequence $X_1,\dots,X_n$ of objects of $\mathbb{M}$ that has a multiproduct $(\gamma_{ij}\colon Y_j\to X_i)_{(i,j)\in[1,n]\times J}$, the map

$$\langle\!\langle F\gamma_{ij}\rangle\!\rangle\colon \coprod_{j\in J} FY_j \to \prod_{i=1}^{n} FX_i$$

is bijective.

If F and H are $\mathbb{M}$-multialgebras, then an $\mathbb{M}$-*homomorphism* from F to H is a natural transformation from F to H.

The category of $\mathbb{M}$-multialgebras and $\mathbb{M}$-homomorphisms is denoted by $\mathbb{M}\mathrm{ulAlg}(\mathbb{M})$.

1.2 Examples.

1.2.0. Algebraic theories and algebras in the sense of F.W. Lawvere [6], and those of I-terms in the sense of J. Benabou [1].

1.2.1 The multialgebraic theory of integral domains. Let $\mathbb{D}_0$ be the category whose objects are pairs (n,I) consisting of an integer $n\in\mathbb{N}$ and a prime ideal I of $\mathbb{Z}[X_1,\dots,X_n]$, and whose morphisms $(n,I)\to(m,J)$ are the injective homomorphisms of unital rings $\mathbb{Z}[X_1,\dots,X_n]/I \to \mathbb{Z}[X_1,\dots,X_m]/J$. These are of the form $\langle g_1,\dots,g_n\rangle$, where $g_1,\dots,g_n$

are polynomials in $\mathbb{Z}[X_1,\ldots,X_m]$ such that $f \in I$ if and only if $f(g_1,\ldots,g_n) \in J$ for all $f \in \mathbb{Z}[X_1,\ldots,X_n]$, and where $\langle g_1,\ldots,g_n\rangle$ denotes the quotient homomorphism of the homomorphism $g_1,\ldots,g_n\colon \mathbb{Z}[X_1,\ldots,X_n] \to \mathbb{Z}[X_1,\ldots,X_m]$. The composition of morphisms is given by composing the ring homomorphisms. The category $\mathbb{D}_0$ has finite multisums, since the family of objects $(0,(p))$, where p runs over all prime numbers, is initial in $\mathbb{D}_0$, and the multisum of (n,I) and (m,J) exists, consisting of the objects $(n+m,K)$, where K runs over the prime ideals of $\mathbb{Z}[X_1,\ldots,X_n,X_{n+1},\ldots,X_{n+m}]$ such that

$$K \cap \mathbb{Z}[X_1,\ldots,X_n] = I \quad \text{and} \quad K \cap \mathbb{Z}[X_{n+1},\ldots,X_{n+m}] = J.$$

We can see that every object of $\mathbb{D}_0$ belongs to a finite multisum of objects of the form $(1,I)$. The opposite category $\mathbb{D}_0^{\mathrm{op}}$ in which we distinguish the objects of the form $(1,I)$ is thus a multialgebraic theory, which we denote by $\mathbb{M}$. *p. 195*

Let A be an integral domain. For each $(x_1,\ldots,x_n) \in A^n$, denote by

$$I_{x_1,\ldots,x_n} = \{P(X_1,\ldots,X_n) \in \mathbb{Z}[X_1,\ldots,X_n] : P(x_1,\ldots,x_n) = 0\}$$

the prime ideal of polynomial relations with coefficients in $\mathbb{Z}$ between the $x_1,\ldots,x_n$. We define a functor $A^{(\,)}\colon \mathbb{M} \to \mathbb{E}\mathrm{ns}$ by

$$A^{(n,I)} = \{(x_1,\ldots,x_n) \in A^n : I_{x_1,\ldots,x_n} = I\}$$

and, for a morphism $\langle g_1,\ldots,g_n\rangle\colon (n,I) \to (m,J)$ of $\mathbb{D}_0$, by

$$A^{\langle g_1,\ldots,g_n\rangle}(x_1,\ldots,x_m) = \big(g_1(x_1,\ldots,x_m),\ldots,g_n(x_1,\ldots,x_m)\big).$$

This functor is an $\mathbb{M}$-multialgebra since we have

$$\coprod_p A^{(0,(p))} \cong 1 \quad \text{and} \quad \coprod_K A^{(n+m,K)} \cong A^{(n,I)} \times A^{(m,J)}$$

(where the first coproduct is over all primes p, and the second coproduct is over all K such that $K \cap \mathbb{Z}[X_1,\ldots,X_n] = I$ and $K \cap \mathbb{Z}[X_{n+1},\ldots,X_{n+m}] = J$). But we can prove that every $\mathbb{M}$-multialgebra is, up to isomorphism, of this form, and thus defines an integral domain. This correspondence is functorial, i.e. if $\mathbb{D}\mathrm{om}$ denotes the category of integral domains and injective homomorphisms, then we can define a functor $V : \mathbb{D}\mathrm{om} \to \mathbb{M}\mathrm{ulAlg}(\mathbb{M})$ by $VA = A^{(\,)}$ and $Vf(x_1,\ldots,x_n) = (f(x_1),\ldots,f(x_n))$. We can, with difficulty, directly prove that V is an equivalence of categories, but this result is also an immediate consequence of Theorem 3.2.

1.2.2 The multialgebraic theory of commutative local rings. Let $\mathbb{L}_0$ be the category whose objects are pairs (n,I) consisting of an integer $n \in \mathbb{N}$ and a prime ideal I of $\mathbb{Z}[X_1,\ldots,X_n]$, and whose morphisms $(n,I) \to (m,J)$ are the homomorphisms of local rings $\mathbb{Z}[X_1,\ldots,X_n]_I \to \mathbb{Z}[X_1,\ldots,X_m]_J$ that are localisations of polynomial rings at prime ideals. These are of the form $[g_1,\ldots,g_n]$, where $g_1,\ldots,g_n$ are polynomials in $\mathbb{Z}[X_1,\ldots,X_n]$ such that $f \in I$ if and only if $f(g_1,\ldots,g_n) \in J$ for all $f \in \mathbb{Z}[X_1,\ldots,X_n]$, and where $[g_1,\ldots,g_n]$ denotes the extension to fractions of the homomorphism $g_1,\ldots,g_n\colon \mathbb{Z}[X_1,\ldots,X_n] \to \mathbb{Z}[X_1,\ldots,X_m]$. The composition of morphisms is given by composing the ring homomorphisms. The category $\mathbb{L}_0$ has finite multisums since the objects of the form $(0,(p))$ form an initial family of objects, and the multisum of (n,I) and (m,J) exists, consisting of objects $(n+m,K)$, where K runs over the prime ideals of $\mathbb{Z}[X_1,\dots,x_n,X_{n+1},\dots,X_{n+m}]$ such that

$$K\cap\mathbb{Z}[X_1,\dots,X_n]=I \quad\text{and}\quad K\cap\mathbb{Z}[X_{n+1},\dots,X_{n+m}]=J.$$

We can see that every object of $\mathbb{L}_0$ belongs to a finite multisum of objects of the form $(1,I)$. The opposite category in which we distinguish the objects of the form $(1,I)$ is thus a multialgebraic theory, which we denote by $\mathbb{M}$.

Let A be a commutative local ring. For every $(x_1,\dots,x_n)\in A^n$, denote by

$$J_{x_1,\dots,x_n}=\{P(X_1,\dots,X_n)\in\mathbb{Z}[X_1,\dots,X_n]:P(x_1,\dots,x_n)\text{ is not invertible}\}$$

the prime ideal. We define a functor $A^{[\,]}\colon\mathbb{M}\to\mathbb{E}\mathrm{ns}$ by p. 196

$$A^{[n,I]}=\{(x_1,\dots,x_n)\in A^n:J_{x_1,\dots,x_n}=I\}$$

and, for a morphism $[g_1,\dots,g_n]\colon(n,I)\to(m,J)$ of $\mathbb{L}_0$, by

$$A^{[g_1,\dots,g_n]}(x_1,\dots,x_m)=\big(g_1(x_1,\dots,x_m),\dots,g_n(x_1,\dots,x_m)\big).$$

This functor is an $\mathbb{M}$-multialgebra since we have

$$\coprod_p A^{[0,(p)]}\cong 1 \quad\text{and}\quad \coprod_K A^{[n+m,K]}\cong A^{[n,I]}\times A^{[m,J]}$$

(where the first coproduct is over all primes p, and the second coproduct is over all K such that $K\cap\mathbb{Z}[X_1,\dots,X_n]=I$ and $K\cap\mathbb{Z}[X_{n+1},\dots,X_{n+m}]=J$). But we can prove that every $\mathbb{M}$-multialgebra is, up to isomorphism, of this form, and thus defines a commutative local ring. This correspondence is functorial, i.e. if $\mathbb{L}\mathrm{occ}$ denotes the category of commutative local rings and local homomorphisms, then we can define a functor $V:\mathbb{L}\mathrm{occ}\to\mathbb{M}\mathrm{ulAlg}(\mathbb{M})$ by $VA=A^{[\,]}$ and $Vf(x_1,\dots,x_n)=(f(x_1),\dots,f(x_n))$. We can, with difficulty, directly prove that V is an equivalence of categories, but this result is also an immediate consequence of Theorem 3.2.

1.2.3 The multialgebraic theory of real pre-Hilbert spaces. A finite sequence of vectors in a real pre-Hilbert space is said to be orthonormal if the vectors are all of norm 1 and pairwise orthogonal. A real matrix with p rows and n columns is said to be orthonormal if its columns form an orthonormal sequence in $\mathbb{R}^p$. Let $\mathbb{O}\mathrm{rth}$ be the category of orthonormal matrices, whose objects are the natural numbers, and whose morphisms $n\to p$ are the orthonormal matrices with p rows and n columns, with composition being given by matrix multiplication. In particular, there is a unique morphism $0\to n$, namely the empty matrix. The category $\mathbb{O}\mathrm{rth}$ is in fact equivalent to the category $\mathbb{E}\mathrm{uc}$ of Euclidean spaces, which is a full subcategory of the category $\mathbb{P}\mathrm{Hilb}$ of real pre-Hilbert spaces. It thus has finite multisums, by [2, 1.1.3]. The opposite category $\mathbb{O}\mathrm{rth}^{\mathrm{op}}$ in which we distinguish the family of objects $(X_\rho)_{\rho\in\mathbb{R}_+}$ defined by $X_0=0$ and $X_\rho=1$ for $\rho>0$ is a multialgebraic theory, which we denote by $\mathbb{M}$.

If E is a real pre-Hilbert space, then we define a functor $E^{(\,)}\colon\mathbb{M}\to\mathbb{E}\mathrm{ns}$ by

$$E^{(n)}=\{(x_1,\dots,x_n)\in E^n:x_1,\dots,x_n\text{ is orthonormal in }E\}$$

and, for a morphism $A = (a_{ji})\colon n \to p$, by

$$E^{(A)}(x_1,\dots,x_p) = (a_{11}x_1 + \dots + a_{p1}x_p, \dots, a_{1n}x_1 + \dots + a_{pn}x_p).$$

We can show that this functor is an $\mathbb{M}$-multialgebra, and, conversely, that every $\mathbb{M}$-multialgebra is, up to isomorphism, of the above form, and thus defines a real pre-Hilbert space. We can thus establish an equivalence between the category $\mathbb{P}$Hilb of real pre-Hilbert spaces and orthogonal linear maps and the category $\mathbb{M}$ulAlg($\mathbb{M}$). This result is an immediate consequence of Theorem 3.2.

1.2.4 The multialgebraic theory of totally ordered sets. Let Δ_{st} be the category whose objects are the finite ordinals, and whose morphisms are the strictly increasing maps. The object 0 is initial, and the category has finite multisums, with the multisum of n and p being given by the set of pairs of morphisms $(f\colon n \to q, g\colon p \to q)$ that are globally surjective. The opposite category $\Delta_{\mathrm{st}}^{\mathrm{op}}$ in which we distinguish the object 1 is a multialgebraic theory, which we denote by $\mathbb{M}$. p. 197

A totally ordered set E determines an $\mathbb{M}$-multialgebra $E^{(\,)}\colon \mathbb{M} \to \mathbb{E}\mathrm{ns}$, defined by

$$E^{(n)} = \left\{(x_1,\dots,x_n) \in E^n : x_1 < x_2 < \dots < x_n\right\}$$

and, for $f\colon n \to p$, by

$$E^{(f)}(x_1,\dots,x_n) = \left(x_{f(1)},\dots,x_{f(n)}\right).$$

Conversely, every $\mathbb{M}$-multialgebra is, up to isomorphism, of the above form, and thus determines a totally ordered set. We thus establish a correspondence between the category $\mathbb{T}$otOrd of totally ordered sets and strictly increasing maps and the category $\mathbb{M}$ulAlg($\mathbb{M}$).

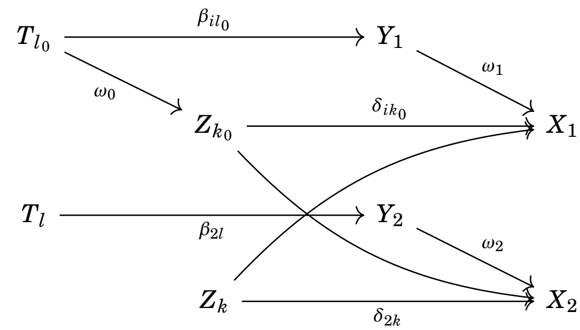
1.3 Proposition. $\mathbb{M}$ulAlg($\mathbb{M}$) *is a multireflexive full subcategory of* $\mathbb{E}\mathrm{ns}^{\mathbb{M}}$ *that is closed under connected limits, filtered colimits, and cokernels of equivalence relations.*

Proof. a) Let $(F_k)_{k\in\mathbb{K}}$ be a connected diagram in $\mathbb{M}$ulAlg($\mathbb{M}$) whose limit in $\mathbb{E}\mathrm{ns}^{\mathbb{M}}$ is F. For every finite multiproduct $(\gamma_{ij}\colon Y_j \to X_i)_{(i,j)\in[1,n]\times J}$ in $\mathbb{M}$, we have [2, 3.5.4.a]

$$\begin{aligned}\coprod_{j\in J} FY_j = \coprod_{j\in J}\varprojlim_{k\in\mathbb{K}} F_kY_j \cong \varprojlim_{k\in\mathbb{K}}\coprod_{j\in J} F_kY_j \cong \varprojlim_{k\in\mathbb{K}}\prod_{i=1}^{n} F_kX_i\\ \cong \prod_{i=1}^{n}\varprojlim_{k\in\mathbb{K}} F_kX_i = \prod_{i=1}^{n} FX_i.\end{aligned}$$

The functor F is thus an $\mathbb{M}$-multialgebra, and so $\mathbb{M}$ulAlg($\mathbb{M}$) is a full subcategory of $\mathbb{E}\mathrm{ns}^{\mathbb{M}}$ that is closed under connected limits. To show that $\mathbb{M}$ulAlg($\mathbb{M}$) is a multireflexive subcategory of $\mathbb{E}\mathrm{ns}^{\mathbb{M}}$ it suffices, by [2, Theorem 3.6.1], to show that the solution-set condition is satisfied.

b) Let G be an $\mathbb{M}$-multialgebra, and $f\colon F \to G$ a morphism in $\mathbb{E}\mathrm{ns}^{\mathbb{M}}$. Since the category $\mathbb{E}\mathrm{ns}^{\mathbb{M}}$ is regular, f factors as $f = gh$, where $h\colon F \to H$ is a quotient functor, and $g\colon H \to G$ is a sub-functor. For every object X of $\mathbb{M}$, denote by $\bar{H}X$ the set of elements of $G(X)$ of the form $G\omega(y)$, where $\omega\colon Y \to X$ runs over the set of morphisms in $\mathbb{M}$ whose source is an object Y belonging to a finite multiproduct $(\gamma_{ij}\colon Y_j \to X_i)_{(i,j)\in[1,n]\times J}$ of objects of the family $(X_g)_{g\in G}$, i.e. $Y = Y_{j_0}$, and where y is an element of GY such that $G\gamma_{ij_0} \in HX_i$ for all $i \in [1,n]$. It is immediate that this defines a sub-functor $\bar{H}$ of G which contains the sub-functor H. We will show that $\bar{H}$ is an $\mathbb{M}$-multialgebra. Let $(\delta_{ik} : Z_k \to X_i)_{(i,k)\in[1,n]\times K}$ be a finite multiproduct in $\mathbb{M}$. The map $\langle\langle(\bar{H}\delta_{ik})\rangle\rangle : \coprod_{k\in K} \bar{H}Z_k \to \prod_{i=1}^n \bar{H}X_i$ induced by the bijection $\langle\langle(G\delta_{ik})\rangle\rangle : \coprod_{k\in K} GZ_k \to \prod_{i=1}^n GX_i$ is injective. To show that it is surjective, consider an element $(x_i) \in \prod_{i=1}^n \bar{H}X_i$, which we will show that the unique element $z \in GZ_{k_0}$ satisfying $G\delta_{ik_0}(z) = x_i$ for all $i \in [1,n]$ belongs to $\bar{H}Z_{k_0}$. p. 198



For all $i \in [1,n]$, the element x_i belongs to $\bar{H}X_i$, and so there exists a morphism $\omega_i : Y_i \to X_i$ and an element $y_i \in GY_i$ satisfying the conditions stated above in the definition of $\bar{H}$. Let $(\beta_{il} : T_l \to Y_i)_{(i,l)\in[1,n]\times L}$ and $t \in GT_{l_0}$ be such that they satisfy $G\beta_{il_0}(t) = y_i$ for all $i \in [1,n]$. The family of morphisms $\omega_i\beta_{il_0} : T_{l_0} \to X_i$ factors uniquely through a family of morphisms $(\delta_{ik_1} : Z_{k_1} \to X_i)_{i\in I}$ and a morphism $\omega_0 : T_{l_0} \to Z_{k_1}$. The relations

$$(G\delta_{ik_1})(G\omega_0(t)) = (G\omega_i)(G\beta_{il_0}(t)) = G\omega_i(y_i) = x_i$$

imply that $k_1 = k_0$ and that $G\omega_0(t) = z$. We can then easily show that $z \in \bar{H}Z_{k_0}$. This proves that $\bar{H}$ is an $\mathbb{M}$-multialgebra. We finally obtain a solution-set of morphisms from F to $\mathbb{M}\mathrm{ulAlg}(\mathbb{M})$ by noting that, up to isomorphism, there exists a set of quotient functors H of F, and thus a set of functors of the form $\bar{H}$.

c) If $(F_k)_{k\in\mathbb{K}}$ is a filtered diagram of $\mathbb{M}\mathrm{ulAlg}(\mathbb{M})$ that has F as its colimit in $\mathbb{E}\mathrm{ns}^{\mathbb{M}}$, then, for every finite multiproduct $(\gamma_{ij} : Y_j \to X_i)_{(i,j)\in[1,n]\times J}$ in $\mathbb{M}$, we have

$$\begin{aligned}\coprod_{j\in J} FY_j = \coprod_{j\in J} \varinjlim_{k\in\mathbb{K}} F_kY_j \cong \varinjlim_{k\in\mathbb{K}} \coprod_{j\in J} F_kY_j \cong \varinjlim_{k\in\mathbb{K}} \prod_{i=1}^n F_kX_i \\ \cong \prod_{i=1}^n \varinjlim_{k\in\mathbb{K}} F_kX_i = \prod_{i=1}^n FX_i\end{aligned}$$

and so F is an $\mathbb{M}$-multialgebra. Thus $\mathbb{M}\mathrm{ulAlg}(\mathbb{M})$ is closed under filtered colimits.

d) Let $(m,n) : R \rightrightarrows F$ be an equivalence relation in $\mathbb{M}\mathrm{ulAlg}(\mathbb{M})$. Let $g : F \to G$ be its cokernel in $\mathbb{E}\mathrm{ns}^{\mathbb{M}}$. Then, for every object X of $\mathbb{M}$, RX is an equivalence relation on FX whose quotient is GX. For every finite multiproduct $(\gamma_{ij} : Y_j \to X_i)_{(i,j)\in[1,n]\times J}$ in $\mathbb{M}$, we have

$$\begin{aligned}\coprod_{j\in J} GY_j \cong \coprod_{j\in J}(FY_j/RY_j) \cong \coprod_{j\in J} FY_j \Big/ \coprod_{j\in J} RY_j \cong \prod_{i=1}^n FX_i \Big/ \prod_{i=1}^n RX_i \\ \cong \prod_{i=1}^n (FX_i/RX_i) \cong \prod_{i=1}^n GX_i.\end{aligned}$$

This proves that G is an $\mathbb{M}$-multialgebra. Thus $\mathbb{M}\text{ulAlg}(\mathbb{M})$ is closed in $\mathbb{E}\text{ns}^{\mathbb{M}}$ under quotients of equivalence relations.

□

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1.4 Proposition. $\mathbb{M}\text{ulAlg}(\mathbb{M})$ *is a multicocomplete category, whose equivalence relations are effective, and whose pairs of coequalisable morphisms have cokernels, and it has a universal factorisation of morphisms into monomorphisms and regular epimorphisms.*

Proof. Let (F_k) be a diagram in $\mathbb{M}\text{ulAlg}(\mathbb{M})$. If $(\iota_k : F_k \to F)_{k\in\mathbb{K}}$ is its colimit in $\mathbb{E}\text{ns}^{\mathbb{M}}$, and if $(f_j : F \to G_j)_{j\in J}$ is a universal family of morphisms from F to $\mathbb{M}\text{ulAlg}(\mathbb{M})$, then it is immediate that $(f_j\iota_k : F_k \to G_j)_{(j,k)\in J\times\mathbb{K}}$ is a multicolimit of $(F_k)_{k\in\mathbb{K}}$ in $\mathbb{M}\text{ulAlg}(\mathbb{M})$. Let $(f,g)\colon F \rightrightarrows G$ be a pair of morphisms in $\mathbb{M}\text{ulAlg}(\mathbb{M})$ that is coequalisable by a morphism $h : G \to H$ in $\mathbb{M}\text{ulAlg}(\mathbb{M})$. Denote by $(k_i : G \to K_i)_{i\in I}$ a multicokernel of (f,g) in $\mathbb{M}\text{ulAlg}(\mathbb{M})$. The set I is non-empty, and so the kernel pair $(m,n)\colon R \rightrightarrows G$ of the family $(k_i : G \to K_i)_{i\in I}$ exists. This is an equivalence relation. Let $k : G \to K$ be its cokernel. Every morphism k_i is of the form $k_i = h_k i$. Consequently, $|I| = 1$, and $k \cong k_i$ is a cokernel of (f,g). The inclusion functor $\mathbb{M}\text{ulAlg}(\mathbb{M}) \to \mathbb{E}\text{ns}^{\mathbb{M}}$ preserves kernel pairs and fibre products. It preserves and reflects regular epimorphisms since these are cokernels of their kernel pairs. Since the category $\mathbb{E}\text{ns}^{\mathbb{M}}$ has universal regular factorisations, so too does the category $\mathbb{M}\text{ulAlg}(\mathbb{M})$.

□

2 Multialgebraic forgetful functors

The *structure-forgetful functor* $U_{\mathbb{M}} : \mathbb{M}\text{ulAlg}(\mathbb{M}) \to \mathbb{E}\text{ns}$ is defined by $U_{\mathbb{M}}F = \coprod_{g\in G} FX_g$ and, for $f : F \to H$ in $\mathbb{M}\text{ulAlg}(\mathbb{M})$, by $U_{\mathbb{M}}f = \coprod_{g\in G} fX_g$.

2.0 Proposition. *The functor* $U_{\mathbb{M}} : \mathbb{M}\text{ulAlg}(\mathbb{M}) \to \mathbb{E}\text{ns}$ *has a left multiadjoint; it is faithful and reflects isomorphisms; it preserves connected limits, filtered colimits, and regular epimorphisms.*

Proof. If we denote by $k_{\mathbb{M}} : \mathbb{M}\text{ulAlg}(\mathbb{M}) \to \mathbb{E}\text{ns}$ the inclusion functor, by $\varphi\colon G \to M$ the functor defined by $\varphi g = X_g$, by $\mathbb{E}\text{ns}^{\varphi} : \mathbb{E}\text{ns}^{\mathbb{M}} \to \mathbb{E}\text{ns}^{G}$ the functor associated to φ, and by $\Sigma : \mathbb{E}\text{ns}^{G} \to \mathbb{E}\text{ns}$ the disjoint sum of sets functor, then we have $U_{\mathbb{M}} = \Sigma\mathbb{E}\text{ns}^{\varphi}k_{\mathbb{M}}$. The functor $k_{\mathbb{M}}$ has a left multiadjoint (Proposition 1.3), and the functor $\mathbb{E}\text{ns}^{\varphi}$ has a left adjoint. The functor Σ also has a left multiadjoint since, for any set E, and writing $\text{Part}(E)$ to mean the set of partitions of E indexed by G, the family of maps

$$\left(1_E : E \to \bigcup_{g\in G} E_g\right)_{(E_g)_{g\in G}\in\text{Part}(E)}$$

is a universal family of morphisms from E to Σ. We thus deduce that the composite functor $U_{\mathbb{M}}$ has a left multiadjoint.

Let $f,h\colon F \rightrightarrows G$ be morphisms in $\mathbb{M}\text{ulAlg}(\mathbb{M})$ such that $Uf = Uh$. Then we have $\coprod_{g\in G} f_{X_g} = \coprod_{g\in G} h_{X_g}$, and so $f_{X_g} = h_{X_g}$ for all $g \in G$. If $(X_i)_{i\in[1,n]}$ is a finite sequence of objects of $(X_g)_{g\in G}$ that has a multiproduct $(\gamma_{ij} : Y_j \to X_i)_{(i,j)\in[1,n]\times J}$, then we have

$$\prod_{i=1}^{n} f_{X_i} = \prod_{i=1}^{n} h_{X_i}$$

and consequently

$$\coprod_{j\in J} f_{Y_j} = \coprod_{j\in J} h_{Y_j}$$

and so $f_{Y_j} = h_{Y_j}$ for all $j \in J$. But, since every object Y of $\mathbb{M}$ is of the above form Y_j, we have $f_Y = h_Y$. Thus $f = h$. p. 200

Let $f: F \to G$ be a morphism in $\mathbb{M}$ulAlg($\mathbb{M}$) such that $U_{\mathbb{M}}f$ is bijective. Then $\coprod_{g\in G} f_{X_g}$ is bijective, and so f_{X_g} is bijective for all $g \in G$. If $(X_i)_{i\in[1,n]}$ is a finite sequence of objects of $(X_g)_{g\in G}$ that has a multiproduct $(\gamma_{ij}: Y_j \to X_i)_{(i,j)\in[1,n]\times J}$, then the map $\prod_{i=1}^{n} f_{X_i}$ is bijective, and so $\coprod_{j\in J} f_{Y_j}$ is bijective, and consequently f_{Y_j} is bijective for all $j \in J$. Since every object Y of $\mathbb{M}$ is of the above form Y_j, we deduce that f_Y is bijective, and consequently that f is an isomorphism.

The functor $U_{\mathbb{M}}$ preserves connected limits since it has a left multiadjoint; it preserves filtered colimits and regular epimorphisms since the functors Σ, $\mathbb{E}\mathrm{ns}^{\varphi}$, and $k_{\mathbb{M}}$ all preserve them. □

2.1 Examples. It is immediate that the structure-forgetful functors $U_{\mathbb{M}}: \mathbb{M}\mathrm{ulAlg}(\mathbb{M}) \to \mathbb{E}\mathrm{ns}$ for the multialgebraic theories $\mathbb{M}$ given in 1.2 are equivalent to the usual structure-forgetful functors.

3 The multialgebraic theory generated by a functor $U: \mathbb{A} \to \mathbb{E}\mathrm{ns}$ that has a left multiadjoint

Let $U: \mathbb{A} \to \mathbb{E}\mathrm{ns}$ be a functor that has a left multiadjoint. For each set E, we choose a universal family of morphisms from E to U, and consider only the morphisms that are diagonally universal to U belonging to these families. We denote by $\mathbb{L}_0$ the full subcategory of $\mathbb{A}$ whose objects are the targets of the diagonally universal morphisms to U whose sources are the finite cardinals.

3.0 Lemma. *The category $\mathbb{L}_0$ is small, has finite multisums, and each of its objects belongs to a finite multisum of objects that are targets of diagonally universal morphisms whose source is the cardinal* 1.

Proof. Let $(X_i)_{i\in[1,n]}$ be a finite sequence of objects of $\mathbb{L}_0$. For every $i \in [1,n]$, let E_i be a finite cardinal, and $g_i: E_i \to UX_i$ a diagonally universal morphism from E_i to U. Set $(E,\iota) = \coprod_{i=1}^{n} E_i$, and denote by $(h_j: E \to UY_j)_{j\in J}$ a universal family of morphisms form E to U. Since the cardinal E is finite, the objects Y_j belong to $\mathbb{L}_0$. Set

$$J' = \{j \in J : h_j\iota_i \text{ factors through } g_i \text{ in the form } h_j\iota_i = (U\gamma_{ji})g_i \text{ for all } i \in [1,n]\}.$$

We will show that $(\gamma_{ji}: X_i \to Y_j)_{(i,j)\in[1,n]\times J'}$ is a multisum of $(X_i)_{i\in[1,n]}$. Let $(f_i: X_i \to Z)_{i\in[1,n]}$ be an inductive cone in $\mathbb{A}$ with base $(X_i)_{i\in[1,n]}$. Then there exists a unique morphism $g: E \to UZ$ such that $g\iota_i = (Uf_i)g_i$ for all $i \in I$. Denote by $h_j: E \to UY_j$ the diagonally universal morphism from E to U, and by $f: Y_j \to Z$ the morphism in $\mathbb{A}$ that satisfies $(Uf)h_j = g$. The relation $(Uf)h_j\iota_i = g\iota_i = (Uf_i)g_i$ implies the existence of a morphism $\gamma_{ji}: X_i \to Y_j$ such that $(U\gamma_{ji})g_i = h_j\iota_i$ and $f\gamma_{ji} = f_i$ for all $i \in [1,n]$. Then

$j \in J'$ and the inductive cone $(f_i : X_i \to Z)_{i\in[1,n]}$ factor uniquely through the inductive cone $(\gamma_{ij} : X_i \to Y_j)_{i\in[1,n]}$. This shows that $\mathbb{L}_0$ has finite multisums.

Let Y be an object of $\mathbb{L}_0$. Then there exists a finite cardinal E and a diagonally universal morphism $h : E \to UY$. The cardinal E is the finite sum of models of the cardinal 1, say $(E, \iota) = \coprod_{i=1}^{n} E_i$, with $E_i = 1$. For all $i \in [1,n]$, the map $h\iota_i : E_i \to UY$ factors through p. 201
a diagonally universal morphism $g_i : E_i \to UX_i$. We then find ourselves in the situation described above. We thus deduce that Y belongs to a multisum of $(X_i)_{i\in[1,n]}$, with each of the X_i being the target of a diagonally universal morphism whose source is the cardinal 1. □

3.1 Notation. The multialgebraic theory $\mathbb{M}$ *generated* by the functor $U : \mathbb{A} \to \mathbb{E}\mathrm{ns}$ is the opposite category of the full subcategory $\mathbb{L}_0$ of $\mathbb{A}$ whose objects are the targets of the diagonally universal morphisms to U whose sources are the finite cardinals, endowed with the distinguished family of the objects that are targets of diagonally universal morphisms whose source is the cardinal 1. Writing $J_0 : \mathbb{L}_0 \to \mathbb{A}$ for the inclusion functor, the *comparison functor* $V : \mathbb{A} \to \mathbb{M}\mathrm{ulAlg}(\mathbb{M})$ is defined by

$$V(\cdot) = \mathrm{Hom}_{\mathbb{A}}(J_0(-), \cdot).$$

It satisfies $U_{\mathbb{M}} V \cong U$. If V is an equivalence, then the functor U is said to be a *multialgebraic forgetful functor*.

3.2 Theorem. *A functor $U : \mathbb{A} \to \mathbb{E}\mathrm{ns}$ is a multialgebraic forgetful functor if and only if*

1) it has a left multiadjoint;

2) it reflects isomorphisms;

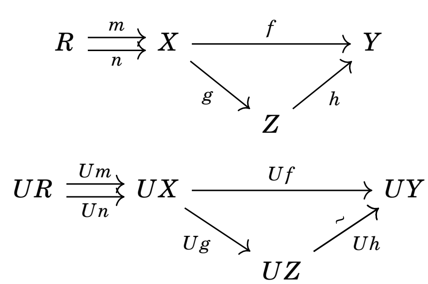
3) $\mathbb{A}$ has filtered colimits and kernel pairs, and its equivalence relations are effective; and

4) it preserves filtered colimits and regular epimorphisms.

Proof. Since the category $\mathbb{A}$ does not necessarily have products, we consider here the kernel pairs of a set of morphisms with the same source. The conditions are necessary by Propositions 1.3, 1.4, and 2.0.

Now consider a functor U satisfying the conditions.

a) We will show that U reflects regular epimorphisms. Let $f : X \to Y$ be a morphism in $\mathbb{A}$ whose image Uf is a surjective map. Denote by $(m,n) : R \rightrightarrows X$ the kernel pair of f, by $g : X \to Z$ the cokernel of (m,n), and by $h : Z \to Y$ the unique morphism such that $hg = f$.

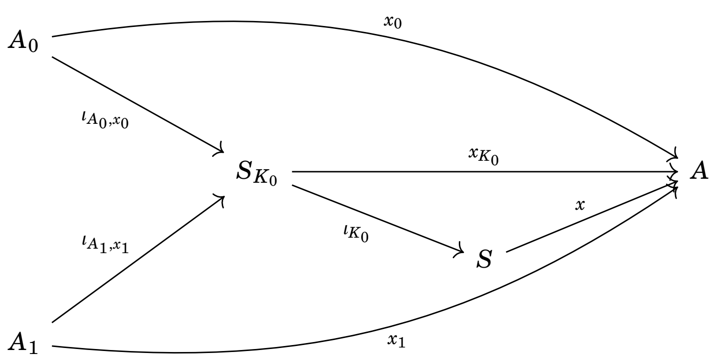


Then (m,n) is the kernel pair of g. Consequently, (Um,Un) is the kernel pair of Uf, and also of Ug. Since Uf and Ug are surjective maps, they are both cokernels of (Um,Un), and so Uh is bijective. Then h is an isomorphism, and $f \cong g$ is a regular epimorphism. *p. 202*

b) We will show that the functor $J_0 \colon \mathbb{L}_0 \to \mathbb{A}$ is dense. We will in fact show that J_0 is dense by filtered colimits and J_0-absolute cokernels [4]. The objects of $\mathbb{L}_0$ are of finite presentation in $\mathbb{A}$, since, if E is a finite cardinal, if $(g_i \colon E \to UA_i)_{i\in I}$ is a universal family of morphisms from E to U, and if $A = \varinjlim_{k\in\mathbb{K}} A_k$ is a filtered colimit in $\mathbb{A}$, then we have

$$\begin{aligned}\coprod_{i\in I} \mathrm{Hom}_{\mathbb{A}}(A_i,A) &\cong \mathrm{Hom}_{\mathbb{E}}\mathrm{ns}(E,UA) \cong \varinjlim_{k\in\mathbb{K}} \mathrm{Hom}_{\mathbb{E}}\mathrm{ns}(E,UA_k)\\ &\cong \varinjlim_{k\in\mathbb{K}} \coprod_{i\in I} \mathrm{Hom}_{\mathbb{A}}(A_i,A_k) \cong \coprod_{i\in I} \varinjlim_{k\in\mathbb{K}} \mathrm{Hom}_{\mathbb{A}}(A_i,A_k)\end{aligned}$$

and so, for all $i \in I$, we have $\mathrm{Hom}_{\mathbb{A}}(A_i,A) \cong \varinjlim_{k\in\mathbb{K}} \mathrm{Hom}_{\mathbb{A}}(A_i,A_k)$. We thus deduce that the filtered colimits of $\mathbb{A}$ are J_0-absolute [4]. Let A be an object of $\mathbb{A}$. Set $K = \{(A_0,x_0) \colon A_0 \in \mathbb{L}_0 \text{ and } x_0 \colon A_0 \to A\}$, and denote by P the set of finite subsets of K. Then P is a filtered ordered set. For $K_0 \in P$, consider a multisum of $(A_0)_{(A_0,x_0)\in K_0}$ in $\mathbb{M}\mathrm{ulAlg}(\mathbb{M})$, and denote by $(\iota_{A_0,x_0} \colon A_0 \to S_{K_0})_{(A_0,x_0)\in K_0}$ the family of morphisms of this multisum, which factors as the family of morphisms $(x_0 \colon A_0 \to A)_{(A_0,x_0)\in K_0}$ and a morphism $x_{K_0} \colon S_{K_0} \to A$.



The objects S_{K_0} are in $\mathbb{L}_0$. For $K_0 \subset K_1 \in P$, we denote by $\iota_{K_1K_0} \colon S_{K_0} \to S_{K_1}$ the canonical morphism. Let $(\iota_{K_0} \colon S_{K_0} \to S)_{K_0\in P}$ be the filtered colimit of $(S_{K_0})_{K_0\in P}$, and $x \colon S \to A$ the morphism defined by $x\iota_{K_0} = x_{K_0}$ for all $K_0 \in P$. For every object $A_0 \in \mathbb{L}_0$, the map $\mathrm{Hom}_{\mathbb{A}}(A_0,x)$ is surjective. Since the map Ux is equivalent to the sum $\coprod_{A_0} \mathrm{Hom}_{\mathbb{A}}(A_0,x)$, where A_0 runs over the targets of the diagonally universal morphisms whose source is 1, it is also surjective. Since the functor U reflects regular epimorphisms, we thus deduce that x is a regular epimorphism. It is thus a J_0-absolute regular epimorphism. We denote by $\mathbb{L}$ the full subcategory of $\mathbb{A}$ whose objects are the filtered colimit of objects of $\mathbb{L}_0$. Every object of $\mathbb{A}$ is then the J_0-absolute regular quotient of an object of $\mathbb{L}$. We thus deduce that every object of $\mathbb{A}$ is the J_0-absolute cokernel of morphisms of $\mathbb{L}$ [4, Lemma 5.6.1], and consequently J_0 is dense by filtered colimit and J_0-absolute cokernels [4, Def. 2.0].

c) The comparison functor $V \colon \mathbb{A} \to \mathbb{M}\mathrm{ulAlg}(\mathbb{M})$ is fully faithful since J_0 is dense; it preserves filtered colimits since the objects of $\mathbb{L}_0$ are of finite presentation in $\mathbb{A}$; and it preserves regular epimorphisms since U preserves them, $U_{\mathbb{M}}$ reflects them, and since we have an isomorphism $U_{\mathbb{M}}V \cong U$. We will show that V is an equivalence of categories. Denote by $\mathbb{L}_{\mathbb{M}}$ the full subcategory of $\mathbb{M}$ulAlg($\mathbb{M}$) whose objects are the filtered colimits of representable $\mathbb{M}$-multialgebras. Every object of $\mathbb{L}_{\mathbb{M}}$ is isomorphic to an object of the form VA, where A is an object of $\mathbb{L}$. Let F be an $\mathbb{M}$-multialgebra. By b) applied to $\mathbb{A} = \mathbb{M}$ulAlg($\mathbb{M}$), F is the regular quotient of an object of $\mathbb{L}_{\mathbb{M}}$. There thus exists an object $A_0 \in \mathbb{L}$ and a regular epimorphism $q_0 \colon VA_0 \to F$.

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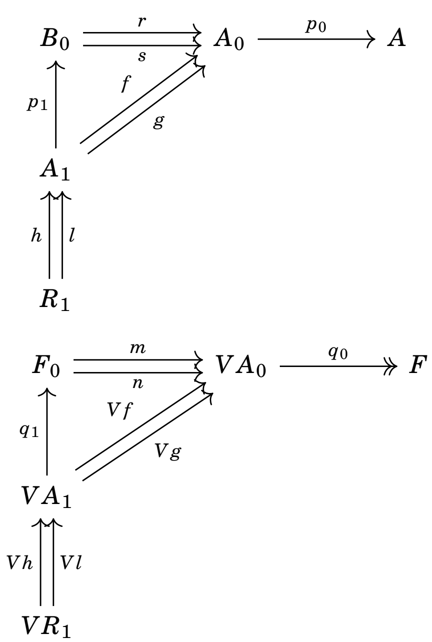

Let $(m,n)\colon F_0 \rightrightarrows VA_0$ be the kernel pair of q_0. There exists, once again, an object A_1 of $\mathbb{L}$ and a regular epimorphism $q_1 \colon VA_1 \to F_0$. Let $f,g\colon A_1 \rightrightarrows A_0$ be morphisms in $\mathbb{A}$ defined by $Vf = mq_1$ and $Vg = nq_1$, let $(h,l)\colon R_1 \rightrightarrows A_1$ be the kernel pair of (f,g), let $p_1\colon A_1 \to B_0$ be the cokernel of (h,l), and let $r,s\colon B_0 \rightrightarrows A_0$ be the morphisms defined by $rp_1 = f$ and $sp_1 = g$. Then (Vh,Vf) is the kernel pair of (Vf,Vg). Since (m,n) is a monomorphic pair, (Vh,Vl) is the kernel pair of q_1. Since V preserves kernel pairs and regular epimorphisms, the morphism Vp_1 is isomorphic to the morphism q_1, and so the pair (Vr,Vs) is isomorphic to the pair (m,n). Since the pair (m,n) is an equivalence, so too is the pair (r,s). It admits a cokernel $p_0\colon A_0 \twoheadrightarrow A$. The two morphisms Up_0 and q_0 are thus isomorphic, and so the object F is isomorphic to VA.

□

4 Multialgebraic categories

A category is *multialgebraic* if it is equivalent to a category $\mathbb{M}$ulAlg($\mathbb{M}$) of multialgebras for some multialgebraic theory $\mathbb{M}$. By §3, it is equivalent to ask that there exist a multialgebraic forgetful functor defined on the category.

4.0 Theorem. *A category is multialgebraic if and only if*

1) it has filtered colimits and kernel pairs, and its equivalence relations are effective;

2) it has finite multisums; and

3) it has a proper generating set consisting of projective objects of finite presentation.

Proof. Recall that an object X is projective if the functor $\mathrm{Hom}(X,-)$ preserves regular epimorphisms, and is of finite presentation if the functor $\mathrm{Hom}(X,-)$ preserves filtered colimits [5]. A category $\mathbb{M}\mathrm{ulAlg}(\mathbb{M})$ satisfies conditions 1), 2), and 3) by taking the generating set to be the set of representable $\mathbb{M}$-multialgebras $\mathrm{Hom}_{\mathbb{M}}(X,-)$, where $X \in \mathbb{M}$. Now let $\mathbb{A}$ be a category satisfying conditions 1), 2), and 3). Let G be a proper generating set of $\mathbb{A}$ consisting of projective objects of finite presentation. Define the functor $U\colon \mathbb{A} \to \mathbb{E}\mathrm{ns}$ by

$$U(-) = \coprod_{A_0 \in G} \mathrm{Hom}_{\mathbb{A}}(A_0,-).$$

We will show that U is a multialgebraic forgetful functor. The functor U preserves filtered colimits since, for a filtered diagram $(A_i)_{i\in\mathbb{I}}$ of $\mathbb{A}$, we have *p. 204*

$$\begin{aligned} U(\varinjlim_{i\in\mathbb{I}} A_i) &\cong \coprod_{A_0\in G} \mathrm{Hom}_{\mathbb{A}}(A_0, \varinjlim_{i\in\mathbb{I}} A_i) \cong \coprod_{A_0\in G} \varinjlim_{i\in\mathbb{I}} \mathrm{Hom}_{\mathbb{A}}(A_0,A_i) \\ &\cong \varinjlim_{i\in\mathbb{I}} \coprod_{A_0\in G} \mathrm{Hom}_{\mathbb{A}}(A_0,A_i) = \varinjlim_{i\in\mathbb{I}} UA_i. \end{aligned}$$

The functor U preserves regular epimorphisms since, for a regular epimorphism f in $\mathbb{A}$, the map $\mathrm{Hom}_{\mathbb{A}}(A_0,f)$ is surjective for all $A_0 \in G$, and so the map $Uf = \coprod_{A_0\in G} \mathrm{Hom}_{\mathbb{A}}(A_0,f)$ is surjective too. The functor U reflects isomorphism since, for a morphism f in $\mathbb{A}$ such that Uf is a bijection, for every $A_0 \in G$, we have that $\mathrm{Hom}_{\mathbb{A}}(A_0,f)$ is a bijection, and so f is an isomorphism. It remains to show that U admits a left multiadjoint. Let I be a set. For every family $(X_i)_{i\in I}$ of objects of G indexed by I, choose a multisum $(\gamma_{ij}\colon X_i \to Y_j)_{(i,j)\in I\times J((X_i))}$ of $(X_i)_{i\in I}$ in $\mathbb{A}$, and for $j \in J((X_i))$ we define the map $g_j\colon I \to \coprod_{A_0\in G} \mathrm{Hom}_{\mathbb{A}}(A_0,Y_j)$ by $g_j(i) = \gamma_{ji}$. We will show that

$$(g_j\colon I \to UY_j)_{j\in\coprod_{(X_i)\in GI} J((X_i))}$$

is a universal family of morphisms from I to U. Let A be an object of $\mathbb{A}$, and let

$$g\colon I \to UA = \coprod_{A_0\in G} \mathrm{Hom}_{\mathbb{A}}(A_0,A)$$

be a map. For $i \in I$, let $X_i \in G$ be such that $g(i) \in \mathrm{Hom}_{\mathbb{A}}(X_i,A)$. We thus obtain an inductive cone $(g(i)\colon X_i \to A)_{i\in I}$ in $\mathbb{A}$ with base $(X_i)_{i\in I}$. There thus exists a unique pair (j,f), where $j \in J((X_i))$ and $f\colon Y_j \to A$ satisfy $f\gamma_{ij} = g(i)$ for all $i \in I$. For $i \in I$, we have $(Uf)g_j(i) = Uf(\gamma_{ji}) = f\gamma_{ji} = g(i)$, and so $(Uf)g_j = g$. Suppose further the existence of another factorisation $g = (Uf')g_{j'}$, where $(X'_i)_{i\in I}$ is a family of objects of G indexed by I, $j' \in J((X_i))$, and $f\colon Y_{j'} \to A$. Then $g(i) \in \mathrm{Hom}_{\mathbb{A}}(X'_i,A)$, so $X'_i = X_i$ for all $i \in I$. Then $f'\gamma_{j'i} = (Uf')g_{j'}(i) = g(i)$, and so $j' = j$ and $f' = f$. □

4.1 Examples. Either Theorem 4.0 or Theorem 3.2 easily show that the following categories are multialgebraic, and that their structure-forgetful functor with values in $\mathbb{E}\mathrm{ns}$ is a multialgebraic forgetful functor.

𝕂	fields and homomorphisms
𝕂c	commutative fields and homomorphisms
𝕂(p)	fields of characteristic p and homomorphisms
𝕂c(0)	commutative fields of characteristic 0 and homomorphisms
𝕃oc	local rings and local homomorphisms
𝕃occ	commutative local rings and local homomorphisms
𝕀nt	integral rings and injective homomorphisms
𝔻om	integral domains and injective homomorphisms
ℝed	reduced commutative rings and injective homomorphisms
ℙrim	primary commutative rings (every zero divisor is nilpotent) and injective homomorphisms

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ℚPrim	quasi-primary commutative rings ($xy = 0$ implies that either x or y is nilpotent) and injective homomorphisms
𝕂dif	differential fields and differential homomorphisms
𝕃ocdif	differential local rings and differential local homomorphisms
𝔻omdif	differential integral domains and injective differential homomorphisms
etc.	

𝕂cO	orderable commutative fields and homomorphisms
𝕃occO	commutative local rings such that $1 + x_1^2 + \ldots + x_n^2$ is invertible for all $x_1, \ldots, x_n$ and local homomorphisms
𝕆rdtot	totally ordered sets and strictly increasing maps
𝕂cord	ordered fields and increasing homomorphisms
𝕃occOrdt	totally ordered commutative local rings and strictly increasing local homomorphisms
𝔻omOrdt	totally ordered integral domains and strictly increasing homomorphisms
etc.	

𝔾rOrd	ordered groups and proper increasing homomorphisms ($f(x) \geqslant 0 \implies x \geqslant 0$)
𝔸bOrd	ordered abelian groups and proper increasing homomorphisms
𝔸ncOrd	ordered commutative rings and proper increasing homomorphisms
etc.	

𝕂cv	commutative fields with absolute values and homomorphisms that preserve the absolute value
etc.	

ℕorm(ℝ)	normed ℝ-vector spaces and linear maps that preserve the norm
𝔸Norm(ℝ)	normed ℝ-algebras and homomorphisms that preserve the norm
𝕊tell(ℂ)	$\mathbb{C}^*$-algebras and homomorphisms that preserve the norm
etc.	

$\mathbb{P}$Hild	pre-Hilbert spaces and orthogonal linear maps (linear maps that preserve the scalar product)
$\mathbb{M}$et	metric spaces and isometries
$\mathbb{T}$rloc	local lattices ($0 \neq 1$ and $[x \vee y = 1 \implies (x = 1 \text{ or } y = 1)]$) and local homomorphisms ($f(x) = 1 \implies x = 1$)
$\mathbb{T}$rdloc	distributive local lattices and local homomorphisms
etc.	

5 Proper multialgebraic functors

5.0 Definitions. If $\mathbb{M}$ and $\mathbb{N}$ are multialgebraic theories, then a *proper morphism* of multialgebraic theories from $\mathbb{N}$ to $\mathbb{M}$ is a functor $m: \mathbb{N} \to \mathbb{M}$ that is bijective on objects and that preserves both the distinguished family of objects and all finite multiproducts. The functor $\mathbb{M}\text{ulAlg}(m): \mathbb{M}\text{ulAlg}(\mathbb{M}) \to \mathbb{M}\text{ulAlg}(\mathbb{N})$ induced by the functor $\mathbb{E}\text{ns}^m: \mathbb{E}\text{ns}^{\mathbb{M}} \to \mathbb{E}\text{ns}^{\mathbb{N}}$ is said to be *proper multialgebraic*. It satisfies *p. 206*

$$U_{\mathbb{N}} \circ \mathbb{M}\text{ulAlg}(m) = U_{\mathbb{M}}.$$

5.1 Theorem. *Every proper multialgebraic functor* $\mathbb{M}\text{ulAlg}(m): \mathbb{M}\text{ulAlg}(\mathbb{M}) \to \mathbb{M}\text{ulAlg}(\mathbb{N})$ *is faithful, reflects isomorphism, preserves filtered colimits and regular epimorphisms, and has a left adjoint.*

Proof. The first properties follow from Proposition 1.3 and from the fact that $\mathbb{M}\text{ulAlg}(m)$ is induced by $\mathbb{E}\text{ns}^m$. Define the functors

$$J_{\mathbb{N}}: \mathbb{N}^{\text{op}} \to \mathbb{M}\text{ulAlg}(\mathbb{N})$$
$$J_{\mathbb{M}}: \mathbb{M}^{\text{op}} \to \mathbb{M}\text{ulAlg}(\mathbb{M})$$

by $J_{\mathbb{N}}(\cdot) = \text{Hom}_{\mathbb{N}}(\cdot, -)$ and $J_{\mathbb{M}}(\cdot) = \text{Hom}_{\mathbb{M}}(\cdot, -)$. Since the functor $J_{\mathbb{N}}$ is dense for filtered colimits and $J_{\mathbb{N}}$-absolute cokernels (part (b) of the proof of Theorem 3.2), and since the category $\mathbb{M}\text{ulAlg}(\mathbb{M})$ has filtered colimits and cokernels of pairs of coequalisable morphisms, the left Kan extension of $J_{\mathbb{M}} m^{\text{op}}$ along $J_{\mathbb{N}}$ exists and determines a left adjoint functor of the functor $\mathbb{M}\text{ulAlg}(m)$ [4, Prop. 3.1]. □

5.2 Examples.

5.2.0. Let $\mathbb{K}_0$ be the category whose objects are pairs (n, I) consisting of a whole number n and a prime ideal I of $\mathbb{Z}[X_1, \ldots, X_n]$, and whose morphisms $(n, I) \to (m, J)$ are field homomorphisms $k(I) \to k(J)$, where $k(I)$ (resp. $k(J)$) denotes the field of fractions of the integral domain $\mathbb{Z}[X_1, \ldots, X_n]/I$ (resp. of $\mathbb{Z}[X_1, \ldots, X_m]/J$). This is a category with finite multisums, calculated as for $\mathbb{D}_0$ (1.2.1). The opposite category $\mathbb{K}_0^{\text{op}}$ in which we distinguish the objects of the form $(1, I)$ is a multialgebraic theory. The category of multialgebras $\mathbb{M}\text{ulAlg}(\mathbb{K}_0^{\text{op}})$ is equivalent to the category $\mathbb{K}$c of commutative fields. A proper morphism of multialgebraic theories $r\colon \mathbb{D}_0^{\mathrm{op}} \to \mathbb{K}_0^{\mathrm{op}}$ is defined by $r(n,I) = (n,I)$ and $r\langle g_1,\dots,g_n\rangle =$ the extension of $\langle g_1,\dots,g_n\rangle$ to fractions. The proper multialgebraic functor

$$\mathbb{M}\mathrm{ulAlg}(r)\colon \mathbb{M}\mathrm{ulAlg}(\mathbb{K}_0^{\mathrm{op}}) \to \mathbb{M}\mathrm{ulAlg}(\mathbb{D}_0^{\mathrm{op}})$$

is equivalent to the inclusion functor $\mathbb{K}\mathrm{c} \to \mathbb{D}\mathrm{om}$, whose left adjoint sends an integral domain to its field of fractions.

5.2.1. The proper morphism of multialgebraic theories $s\colon \mathbb{L}_0^{\mathrm{op}} \to \mathbb{K}_0^{\mathrm{op}}$ is the identity on objects, and sends $[g_1,\dots,g_n]\colon \mathbb{Z}[X_1,\dots,X_n]_I \to \mathbb{Z}[X_1,\dots,X_m]_J$ to the quotient homomorphism $s[g_1,\dots,g_n]\colon k(I) \to k(J)$. The proper multialgebraic functor $\mathbb{M}\mathrm{ulAlg}(s)\colon \mathbb{M}\mathrm{ulAlg}(\mathbb{K}_0^{\mathrm{op}}) \to \mathbb{M}\mathrm{ulAlg}(\mathbb{L}_0^{\mathrm{op}})$ is equivalent to the inclusion functor $\mathbb{K}\mathrm{c} \to \mathbb{L}\mathrm{occ}$, whose left adjoint sends a commutative local ring to its quotient by its maximal ideal.

5.2.2. Let $\mathbb{P}_0$ be the category whose objects are pairs (n,I) consisting of a whole number n and a prime ideal I of $\mathbb{Z}[X_1,\dots,X_n]$, and whose morphisms $(n,I) \to (m,J)$ are the homomorphisms of unital rings $f\colon \mathbb{Z}[X_1,\dots,X_n] \to \mathbb{Z}[X_1,\dots,X_m]$ such that $f^{-1}(J) = I$. This is a category with finite multisums, calculated as for $\mathbb{D}_0$. The opposite category $\mathbb{P}_0^{\mathrm{op}}$ in which we p. 207 distinguish the objects of the form $(1,I)$ is a multialgebraic theory. The category of multialgebras $\mathbb{M}\mathrm{ulAlg}(\mathbb{P}_0^{\mathrm{op}})$ is equivalent to the category $\mathbb{A}\mathrm{nc/Spec}$ whose objects are pairs (A,P) consisting of a commutative unital ring A and a prime ideal P of A, and whose morphisms $(A,P) \to (B,Q)$ are the homomorphisms of unital rings $g\colon A \to B$ such that $g^{-1}(Q) = P$. We define a proper morphism of multialgebraic theories $t\colon \mathbb{P}_0^{\mathrm{op}} \to \mathbb{D}_0^{\mathrm{op}}$ by $t(n,I) = (n,I)$, and by $t(f) =$ quotient of f. The proper multialgebraic functor $\mathbb{M}\mathrm{ulAlg}(t)\colon \mathbb{M}\mathrm{ulAlg}(\mathbb{D}_0^{\mathrm{op}}) \to \mathbb{M}\mathrm{ulAlg}(\mathbb{P}_0^{\mathrm{op}})$ is equivalent to the functor $\mathbb{D}\mathrm{om} \to \mathbb{A}\mathrm{nc/Spec}$ that sends an integral domain A to the pair $(A,\{0\})$, and whose left adjoint sends a pair (A,P) to the integral domain A/P.

5.2.3. The proper morphism of multialgebraic theories $u\colon \mathbb{P}_0^{\mathrm{op}} \to \mathbb{L}_0^{\mathrm{op}}$ is defined by $u(n,I) = (n,I)$, and by $u(f) =$ localisation of f. The functor

$$\mathbb{M}\mathrm{ulAlg}(u)\colon \mathbb{M}\mathrm{ulAlg}(\mathbb{L}_0^{\mathrm{op}}) \to \mathbb{M}\mathrm{ulAlg}(\mathbb{P}_0^{\mathrm{op}})$$

is equivalent to the functor $\mathbb{L}\mathrm{occ} \to \mathbb{A}\mathrm{nc/Spec}$ that sends a local ring A to the pair (A,M_A), where M_A is the maximal ideal of A, and whose left adjoint sends a pair (A,P) to the localised ring A_P.

5.2.4. The proper morphism of multialgebraic theories $rt = su\colon \mathbb{P}_0^{\mathrm{op}} \to \mathbb{K}_0^{\mathrm{op}}$ defines the functor $\mathbb{M}\mathrm{ulAlg}(rt)\colon \mathbb{K}\mathrm{c} \to \mathbb{A}\mathrm{nc/Spec}$ that sends a commutative field K to the pair $(K,\{0\})$, and whose left adjoint sends a pair (A,P) to the field $k(P)$.

6 Multimonadic categories of finite rank

Multimonads and multimonadic categories are defined in [3].

6.0 Definitions. A multimonad $(S,\mathbb{T}) = (S;(T,\eta,\mu))$ on $\mathbb{E}\mathrm{ns}$ is of *finite rank* if the functor S preserves cofiltered limits and the functor T preserves filtered colimits. A category equivalent to $\mathbb{E}\mathrm{ns}^{\mathbb{T}}_{/S}$ and a functor equivalent to $U^{\mathbb{T}}_S\colon \mathbb{E}\mathrm{ns}^{\mathbb{T}}_{/S} \to \mathbb{E}\mathrm{ns}$ are said to be *multimonadic of finite rank* over $\mathbb{E}\mathrm{ns}$.

6.1 Theorem. *For a functor $U\colon \mathbb{A} \to \mathbb{E}\mathrm{ns}$, the following claims are equivalent:*

(i) U is multimonadic of finite rank;

(ii) U is multimonadic and $\mathbb{A}$ has filtered colimits preserved by U; and

(iii) U is a multialgebraic forgetful functor.

Proof. The equivalence (ii) $\iff$ (iii) follows immediately from Theorem 3.2 and from [3, Theorem 4.2]. We will show the equivalence (i) $\iff$ (ii) for $U_S^{\mathbb{T}}\colon \mathbb{E}\mathrm{ns}_{/S}^{\mathbb{T}} \to \mathbb{E}\mathrm{ns}$. First of all, it is immediate that the category $\mathbb{E}\mathrm{ns}_{/S}$ has filtered colimits preserved by U_S if and only if the functor S preserves cofiltered limits. If we assume (i), then $\mathbb{E}\mathrm{ns}_{/S}$ has filtered colimits preserved by U_S, and so $\mathbb{E}\mathrm{ns}_{/S}^{\mathbb{T}}$ has filtered colimits preserved by U, and so $\mathbb{E}\mathrm{ns}_{/S}^{\mathbb{T}}$ has filtered colimits preserved by $U_S^{\mathbb{T}}$. Now assume (ii). Let $(X_i)_{i\in\mathbb{I}}$ be a filtered diagram in $\mathbb{E}\mathrm{ns}_{/S}$. Denote by $(\iota_i\colon U_S X_i \to E)_{i\in\mathbb{I}}$ the colimit of $(U_S X_i)_{i\in\mathbb{I}}$ in $\mathbb{E}\mathrm{ns}$, and by $(l_i\colon F^{\mathbb{T}}X_i \to (Y,y))_{i\in\mathbb{I}}$ the colimit of $(F^{\mathbb{T}}X_i)_{i\in\mathbb{I}}$ in $\mathbb{E}\mathrm{ns}_{/S}^{\mathbb{T}}$. Since the functor $U_S^{\mathbb{T}}$ preserves filtered colimits, p. 208

$$(U_S^{\mathbb{T}} l_i\colon U_S^{\mathbb{T}} F^{\mathbb{T}} X_i \to U_S Y)_{i\in\mathbb{I}}$$

is a colimit of $(U_S^{\mathbb{T}} F^{\mathbb{T}} X_i)_{i\in\mathbb{I}}$. The maps

$$(U_s \eta_{X_i}\colon U_S X_i \to U_S^{\mathbb{T}} F^{\mathbb{T}} X_i)_{i\in\mathbb{I}}$$

determine, by colimits, a map $p\colon E \to U_S Y$ such that $p\iota_i = (U_S^{\mathbb{T}} l_i)(U_S\eta_i)$ for all $i\in\mathbb{I}$. There thus exists a unique object X of $\mathbb{E}\mathrm{ns}_{/S}$ and a unique morphism $\eta\colon X \to Y$ such that $U_S\eta = p$, and a unique diagram $(\gamma_i\colon X_i \to X)_{i\in I}$ such that $(X_i)_{i\in\mathbb{I}}$. Let $(f_i\colon X_i \to Z)_{i\in\mathbb{I}}$ be an inductive cone with base $(X_i)_{i\in\mathbb{I}}$. Then there exists a unique morphism $g\colon E \to U_S Z$ such that $g\iota_i = U_s f_i$ for all $i \in I$. Let $f\colon X' \to Z$ be the unique morphism such that $U_S f = g$. Then

$$\begin{aligned} U_S(\eta_Z f)\iota_i &= (U_S\eta_Z)(U_S f)\iota_i = (U_S\eta_Z)(U_S f_i) = U_S(\eta_Z f_i) = U_S((U^{\mathbb{T}}F^{\mathbb{T}} f_i)\eta_{x_i}) \\ &= (U_S U^{\mathbb{T}}(l l_i))(U_S\eta_{x_i}) = (U_S^{\mathbb{T}} l)(U_S^{\mathbb{T}} l_i)(U_S\eta_{x_i}) = (U_S^{\mathbb{T}})p\iota_i \\ &= U_S((U^{\mathbb{T}} l)\eta)\iota_i. \end{aligned}$$

We thus deduce the equality $(U^{\mathbb{T}} l)\eta = \eta_Z f$, and so f is a morphism $X \to Z$ such that $f\gamma_i = f_i$ for all $i \in I$. Thus $\mathbb{E}\mathrm{ns}_{/S}$ has filtered colimits preserved by U_S. Since the functor U_S reflects isomorphisms, it also reflects filtered colimits, and since U preserves filtered colimits, $U^{\mathbb{T}}$ also preserves filtered colimits. We thus deduce that S preserves cofiltered limits, and that T preserves filtered colimits, i.e. that $(S,\mathbb{T})$ is of finite rank. □

7 α-multialgebraic theories and categories

We consider a regular infinite cardinal α (say $\alpha = \aleph_0$, $\alpha = \aleph_1$, ...). A family is α-small if its index set has cardinality less than α. A multiproduct of an α-small family of objects is said to be α-small. A category has *α-small multiproducts* if every α-small family of objects has a multiproduct.

7.0 Definitions. An *α-multialgebraic theory* is a small category $\mathbb{M}$ with α-small multiproducts endowed with a small distinguished family $(X_g)_{g\in G}$ of objects such that every object of $\mathbb{M}$ belongs to an α-small multiproduct of objects of this family.

An $\mathbb{M}$-*multialgebra* is then a functor $F : \mathbb{M} \to \mathbb{E}\mathrm{ns}$ that is multicontinuous for α-small multiproducts [2].

The category $\mathbb{M}\mathrm{ulAlg}(\mathbb{M})$ is defined as before.

A *proper morphism of α-multialgebraic theories* is a functor that is bijective on objects and that preserves both the distinguished family of objects and all α-small multiproducts.

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7.1. All the above results remain true if we substitute:

- "α-small" for "finite";
- "α-filtered" for "filtered";
- "α-presentable" for "of finite presentation"; and
- "rank-α" for "finite rank".

7.2 Examples of $\aleph_1$-multialgebraic categories.

$\mathbb{M}$etcompl	complete metric spaces and isometries
$\mathbb{M}$etcomp	compact metric spaces and isometries
$\mathbb{B}$an($\mathbb{R}$)	real Banach spaces and linear maps that preserve the norm
$\mathbb{A}$lgBan($\mathbb{R}$)	real Banach algebras and homomorphisms that preserve the norm
$\mathbb{H}$ilb	Hilbert spaces and orthogonal linear maps

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