

# Notes on the Riemann $\zeta$ function, 2

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## Translator's note.

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*What follows is a translation of the French paper:*

BALAZARD, M., SAIAS, E., AND YOR, M. "Notes sur la fonction  $\zeta$  de Riemann, 2". *Advances in Mathematics*, Volume **143** (1999), 284–287.

| p. 284

We denote by  $\sum_{\Re \rho > 1/2}$  a sum over the possible zeros of  $\zeta(s)$  with real part greater than  $\frac{1}{2}$ , where the zeros of multiplicity  $m$  are counted  $m$  times. The goal of this note is the proof of the following result.

**Theorem.** *We have*

$$\frac{1}{2\pi} \int_{\Re(s)=1/2} \frac{\log |\zeta(s)|}{|s|^2} |ds| = \sum_{\Re \rho > 1/2} \log \left| \frac{\rho}{1-\rho} \right|. \quad (1)$$

*In particular, the Riemann hypothesis is true if and only if*

$$\int_{\Re(s)=1/2} \frac{\log |\zeta(s)|}{|s|^2} |ds| = 0.$$

*Proof.* This proof consists of two steps.

**First step.** We start by stating some properties satisfied by a generic function  $f$  in the Hardy space  $H^p(\mathbf{D})$ , where  $\mathbf{D} = \{z \in \mathbb{C} : |z| < 1\}$ , and  $p$  is a positive real number. We denote by  $f^*$  the function defined almost everywhere on the trigonometric circle  $\partial\mathbf{D} = \{z \in \mathbb{C} : |z| = 1\}$  by  $f^*(e^{i\theta}) = \lim_{r \rightarrow 1^-} f(re^{i\theta})$ . We use the letter  $z$  to denote an element of the trigonometric disc  $\mathbf{D}$ , and write

$$s = s(z) = \frac{1}{2} + \frac{1+z}{2(1-z)} = \frac{1}{1-z}.$$

This formula defines a conformal representation of the disc  $\mathbf{D}$  in the semi-plane  $\Re(s) > 1/2$ . | p. 285

\*<https://thosgood.com/translations>

By Jensen's formula (see, for example, [4, Theorem 3.61]), we have, for  $f(0) \neq 0$  and  $r < 1$ ,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{i\theta})| d\theta = \log |f(0)| + \sum_{\substack{|\alpha| < r \\ f(\alpha)=0}} \log \frac{r}{|\alpha|} \quad (2)$$

where, in the sum, the zeros of multiplicity  $m$  are counted  $m$  times. Denote by

$$\exp \left\{ - \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta) \right\}$$

the singular interior factor of  $f$ . As  $r$  tends to 1, Equation (2) becomes (cf. [2, p. 68])

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{i\theta})| d\theta = \log |f(0)| + \sum_{\substack{|\alpha| < 1 \\ f(\alpha)=0}} \log \frac{1}{|\alpha|} + \int_{-\pi}^{\pi} d\mu(\theta). \quad (3)$$

This formula is a consequence of the factorisation theorem for functions in  $H^p$ ; it is stated in [2] for  $p = 1$ , but also holds for all positive values of  $p$ .

Second step. Now consider the function

$$f(z) = (s-1)\zeta(s)$$

(where  $s = 1/(1-z)$ ). The elementary properties of the Riemann  $\zeta$  function (see, for example, [5]) allow us to show that, on one hand,  $f$  belongs to the Hardy space  $H^{1/3}(\mathbf{D})$ , and, on the other hand, that the measure  $\mu$  associated to the singular interior factor of  $f$  is zero (for this latter point, it suffices to reuse the argument developed by Bercovici and Foias for the interior factor of the functions  $(\theta - \theta^s)\zeta(s)(s + 1/2)/s$ , found in the proof of [1, Proposition 2.1]). We can equally show that

$$\begin{aligned} \int_{-\pi}^{\pi} \log |f^*(e^{i\theta})| d\theta &= \int_{\Re(s)=1/2} \frac{\log |\zeta(s)|}{|s|^2} |ds|, \\ \log |f(0)| &= 0, \\ \sum_{\substack{|\alpha| < 1 \\ f(\alpha)=0}} \log \frac{1}{|\alpha|} &= \sum_{\Re(\rho) > 1/2} \log \left| \frac{\rho}{1-\rho} \right|. \end{aligned}$$

With all this information, our result follows from Equation (3).

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□

We finish with some remarks. There are statements related to ours in the works [7, 6] of Wang and Volchkov. It is even possible that Jensen himself was aware of Equation (1) (the reader can consult the article [3] where Jensen informs Mittag-Leffler of his discovery of Equation (2)). It seem interesting, however, to present things as we have done here, and this is for the following three reasons:

- (a) Equation (1) is simpler than those that appear in [7, 6];
- (b) we show here that, to establish Equation (1), it is natural to place ourselves in the framework of Hardy spaces;

- (c) the form of the integral in [Equation \(1\)](#) allows us to interpret this result via Brownian motion, as we show below.

Denote by  $Z = X + iY$  the planar Brownian motion from 0 (or from 1), and by  $Z_{T_{1/2}} = \frac{1}{2} + iY_{T_{1/2}}$  its first point of impact on the critical line  $\Re s = 1/2$ , where  $T_{1/2} := \inf\{t : X_t = 1/2\}$ . We know that  $Y_{T_{1/2}}$  follows a Cauchy law with parameter 1/2. In other words, the law of  $Y_{T_{1/2}}$  has density  $1/2\pi(1/4 + t^2)$ . Thus the second part of the theorem can be stated in the following manner: the Riemann hypothesis is true if and only if

$$\mathbb{E}[\log|\zeta(Z_{T_{1/2}})|] = 0.$$

## Thanks

We thank Luis Báez-Duarte, Michel Delasneri, Catherine Donati, Laurent Habsieger, Aleksandar Ivić, and Alain Plagne for useful conversations.

## References

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