The primary obstruction to deformation

Adrien Douady

21st of November, 1960

Translator's note.

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What follows is a translation of the French seminar talk:

DOUADY, A. "Obstruction primaire à la déformation". *Séminaire Henri Cartan*, Volume **13 (1)** (1960–1961), Talk no. 4. http://www.numdam.org/item/SHC_1960-1961__13_1_A3_0

Contents

Introduction

p. 4-01

Let V_0 be a compact complex-analytic manifold, and let Θ be the sheaf of germs of holomorphic fields of tangent vectors. We ask the following question: given an element $\alpha \in \mathrm{H}^1(V_0,\Theta)$, does there exists a deformation of V_0, with a non-singular base (i.e. a fibred mixed manifold $\pi\colon V \to B$, with $b_0 \in B$, along with an isomorphism $V_0 \xrightarrow{\cong} \pi^{-1}(b_0)$), such

[*] https://thosgood.com/translations

that α is the image, under the map ρ defined in [Talk no. 2], of a vector v that is tangent to B at b_0? An element $\alpha \in \mathrm{H}^1(V_0,\Theta)$ for which the answer is positive is called a *deformation vector*. We will give a necessary condition for α to be a deformation vector; this condition is written $[\alpha \smile \alpha] = 0$. We will then give an example where this condition is not satisfied.

I Exact sequences of sheaves of algebras

Let K be a commutative ring, and let Φ, Φ_1, and Φ_2 be sheaves of K-modules on some space X, and suppose that we have some given homomorphism $\Phi_1 \otimes \Phi_2 \to \Phi$, written as a product. We define, for any cover $\mathcal{U}$ of X, the *cup product*

$$\smile : C^p(X,\mathcal{U};\Phi_1) \otimes C^q(X,\mathcal{U};\Phi_2) \to C^{p+q}(X,\mathcal{U};\Phi)$$

by the formula

$$(\alpha \smile \beta)_{i_0,\ldots,i_{p+q}} = \alpha_{i_0,\ldots,i_p} \cdot \beta_{i_p,\ldots,i_{p+q}}.$$

We have the relation

$$\mathrm{d}(\alpha \smile \beta) = \mathrm{d}\alpha \smile \beta + (-1)^p \alpha \smile \mathrm{d}\beta.$$

This induces a cup product on the cohomology of the cover $\mathcal{U}$, and, by passing to the inductive limit over open covers, a cup product

$$\smile : \mathrm{H}^p(X;\Phi_1) \otimes \mathrm{H}^q(X;\Phi_2) \to \mathrm{H}^{p+q}(X;\Phi).$$

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Definition. A *sheaf of algebras* on X is a sheaf of modules Φ on X endowed with a product $\Phi \otimes \Phi \to \Phi$ (which we do not assume to be either commutative nor associative).

If $f : \Phi \to \Psi$ is a homomorphism of sheaves of algebras, then the kernel Φ' of f is a sheaf of two-sided ideals of Φ, i.e. we have products $\Phi' \otimes \Phi \to \Phi'$ and $\Phi \otimes \Phi' \to \Phi'$ such that the two diagrams

$$\begin{array}{ccc} \Phi' \otimes \Phi & \longrightarrow & \Phi' \\ \downarrow & & \downarrow \\ \Phi \otimes \Phi & \longrightarrow & \Phi \end{array} \qquad \begin{array}{ccc} \Phi \otimes \Phi' & \longrightarrow & \Phi' \\ \downarrow & & \downarrow \\ \Phi \otimes \Phi & \longrightarrow & \Phi \end{array}$$

both commute.

Proposition 1. *Let $0 \to \Phi' \to \Phi \to \Phi'' \to 0$ be an exact sequence of sheaves of algebras on X; let $a \in \mathrm{H}^p(X;\Phi'')$. Then $\delta a \in \mathrm{H}^{p+1}(X;\Phi')$, and, for any class $b \in \mathrm{H}^q(X;\Phi')$, we have*

$$\delta a \smile b = 0.$$

Proof. Let $\mathcal{U}$ be a cover of X such that a and b are represented by cocycles α and β (respectively), and such that α lifts to a cochain $\eta \in C^p(X,\mathcal{U};\Phi)$. Then $\delta\eta$ is a cocycle in $C^{p+1}(X,\mathcal{U};\Phi')$ whose class in $\mathrm{H}^{p+1}(X;\Phi')$ is, by definition, δa, and $\delta a \smile b$ is the class of $\delta\eta \smile \beta$. But $\delta(\eta \smile \beta) = \delta\eta \smile \beta$, and $\eta \smile \beta$ is a cochain in $C^{p+q}(X,\mathcal{U};\Phi')$, since Φ' is a sheaf of ideals. So the cocycle $\delta\eta \smile \beta$ is cohomologous to 0 in $\mathrm{H}^{p+q+1}(X;\Phi')$, which proves the proposition. □

II The primary obstruction

Let V_0 be a complex-analytic manifold, and Θ_0 the sheaf of germs of holomorphic fields of tangent vectors. Then Θ_0 is a sheaf of Lie algebras, and, if $a, b \in \mathrm{H}^\bullet(V_0, \Theta_0)$, then we denote by $[a \smile b]$ the cup product defined by the bracket $[-,-]\colon \Theta_0 \otimes \Theta_0 \to \Theta_0$. It satisfies

$$[b \smile a] = (-1)^{pq+1}[a \smile b]$$

for $a \in \mathrm{H}^p(V_0, \Theta_0)$ and $b \in \mathrm{H}^q(V_0, \Theta_0)$. *p. 4-03*

Theorem 1. *Let $\pi\colon V \to B$ be a mixed manifold, b_0 a point of B, $V_0 = \pi^{-1}(b_0)$, and let $\rho_0\colon T_0 \to \mathrm{H}^1(V_0, \Theta_0)$ be Spencer–Kodaira map. Then, if u and v are tangent vectors of B at b_0, we have*

$$[\rho_0(u) \smile \rho_0(v)] = 0.$$

Corollary. *Let V_0 be a complex-analytic manifold, and Θ the sheaf of germs of holomorphic fields of tangent vectors of V_0. If $a \in \mathrm{H}^1(V_0, \Theta)$ is a deformation vector, then*

$$[a \smile a] = 0.$$

Proof of the Corollary. This is simply a particular case of Theorem 1; note that $[a \smile b]$ is a symmetric bilinear map from $\mathrm{H}^1 \otimes \mathrm{H}^1$ to H^2, and that we are in characteristic $0 \neq 2$. ☐

Proof of Theorem 1. Consider the following sheaves on V_0:

- Θ_0: the sheaf of germs of vertical holomorphic fields on V_0 ;
- $\widetilde{\Theta}_0$: the sheaf of germs of vertical holomorphic fields on V ;
- Π_0: the sheaf of germs of locally projectable holomorphic fields on V_0 ;
- $\widetilde{\Pi}_0$: the sheaf of germs of locally projectable holomorphic fields on V ;
- Λ_0: the sheaf $\pi^* T_0$, where T_0 is the tangent space of B at b_0 ; and
- $\widetilde{\Lambda}_0$: the sheaf $\pi^* \widetilde{T}_0$, where $\widetilde{T}_0$ is the space of germs at b_0 of fields on B of tangent vectors of B.

We have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{\Theta}_0 & \longrightarrow & \widetilde{\Pi}_0 & \longrightarrow & \widetilde{\Lambda}_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\epsilon} & & \downarrow{\scriptstyle\epsilon} & & \downarrow{\scriptstyle\epsilon} & & \\ 0 & \longrightarrow & \Theta_0 & \longrightarrow & \Pi_0 & \longrightarrow & \Lambda_0 & \longrightarrow & 0 \end{array}$$

whence we obtain the following commutative diagram: *p. 4-04*

$$\begin{array}{ccc} \widetilde{T}_0 & \xrightarrow{\widetilde{\rho}} & \mathrm{H}^1(V_0,\widetilde{\Theta}_0) \\ {\scriptstyle\epsilon}\downarrow & & \downarrow{\scriptstyle\epsilon} \\ T_0 & \xrightarrow[\rho]{} & \mathrm{H}^1(V_0,\Theta_0) \end{array}$$

Let $u,v \in T_0$ be fixed tangent vectors of B at b_0. We can always find vector fields $\widetilde{u}$ and $\widetilde{v}$ on B that take the values u and v (respectively) at b_0; $\epsilon(\widetilde{u}) = u$ and $\epsilon(\widetilde{v}) = v$. The exact sequence

$$0 \to \widetilde{\Theta}_0 \to \widetilde{\Pi}_0 \to \widetilde{\Lambda}_0 \to 0$$

is a sequence of homomorphisms of sheaves of Lie algebras, and so

$$[\widetilde{\rho}(\widetilde{u}) \smile \widetilde{\rho}(\widetilde{v})] = 0$$

by Proposition 1. But $\epsilon\colon \widetilde{\Theta}_0 \to \Theta_0$ is also a homomorphism of sheaves of Lie algebras, and the diagram

$$\begin{array}{ccc} \mathrm{H}^1(V_0,\widetilde{\Theta}_0) \otimes \mathrm{H}^1(V_0,\widetilde{\Theta}_0) & \xrightarrow{[-\smile-]} & \mathrm{H}^2(V_0,\widetilde{\Theta}_0) \\ {\scriptstyle\epsilon\otimes\epsilon}\downarrow & & \downarrow{\scriptstyle\epsilon} \\ \mathrm{H}^1(V_0,\Theta_0) \otimes \mathrm{H}^1(V_0,\widetilde{\Theta}_0) & \xrightarrow[{[-\smile-]}]{} & \mathrm{H}^2(V_0,\Theta_0) \end{array}$$

commutes. We thus deduce that $[\rho(u) \smile \rho(v)] = 0$. □

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Remarks.

1. We make essential use of the fact that $\epsilon\colon \widetilde{T}_0 \to T_0$ is surjective, and thus of the fact that B has no singularities.

2. We actually have $[\rho(u) \smile b] = 0$ for all $u \in T_0$, for any class $b \in \mathrm{H}^1(V_0,\Theta_0)$ that is in the image of $\mathrm{H}^1(V_0,\widetilde{\Theta}_0)$ under ϵ. In particular, for an element $a \in \mathrm{H}^1(V_0,\Theta_0)$ to be a regular deformation vector (in the sense of [Talk no. 3]), it is necessary and sufficient for $[a \smile b] = 0$ for all $b \in \mathrm{H}^1(V_0,\Theta_0)$.

If V_0 is a compact complex-analytic manifold, and $a \in \mathrm{H}^1(V_0,\Theta)$, then we call $[a \smile a] \in \mathrm{H}^2(V_0,\Theta)$ the *primary obstruction* to the deformation of V_0 along a. For a to be a deformation vector, it is necessary that this primary obstruction be zero; but it is not sufficient: we can define a sequence of set-theoretic maps ω_n, called *obstructions*, with $\omega_1\colon \mathrm{H}^1(V_0,\Theta) \to \mathrm{H}^2(V_0,\Theta)$ given by $\omega_1(a) = [a \smile a]$, and with ω_{k+1} defined on the subset of $\mathrm{H}^1(V_0,\Theta)$ where ω_k vanishes, with values in varying quotients[1] of $\mathrm{H}^2(V_0,\Theta)$, and a necessary condition for a to be a deformation vector is that all the $\omega_k(a)$ be defined and real. I do not know if *this* condition is sufficient. Kodaira, Spencer, and Nijenhuis [4] have shown that, if $\mathrm{H}^2(V_0,\Theta) = 0$, then every element of $\mathrm{H}^1(V_0,\Theta)$ is a deformation vector. In this case, we even have a locally universal deformation whose base is a manifold, and ρ is an isomorphism from the tangent space of this manifold to $\mathrm{H}^1(V_0,\Theta)$

[1] See the appendix.

III An example of obstruction

1 The manifold V_0

Let $X = E/\Gamma$ be a 2-dimensional complex torus, i.e. $E \cong \mathbb{C}^2$ and $\Gamma \cong \mathbb{Z}^4$, and let D the be projective line $\mathbb{P}^1\mathbb{C}$. Set $V_0 = X \times D$. The sheaf Θ of holomorphic fields of tangent vectors of V_0 is the direct sum of the sheaves of Lie algebras Θ_1 and Θ_2, where

$$\Theta_1 = \mathscr{O} \otimes_{\mathscr{O}_X} \pi_1^* \Theta_X$$
$$\Theta_2 = \mathscr{O} \otimes_{\mathscr{O}_D} \pi_2^* \Theta_D$$

where $\pi_1 \colon V_0 \to X$ and $\pi_2 \colon V_0 \to D$ are the projections, $\mathscr{O}$, $\mathscr{O}_X$, and $\mathcal{D}$ are the structure sheaves (sheaves of local rings), and Θ_X and Θ_D are the sheaves of germs of holomorphic fields of tangent vectors of X and D (respectively). We are mostly interested in Θ_2. Also, $\mathrm{H}^1(V_0, \Theta_2)$ is given by the Künneth exact sequence: p. 4-06

$$0 \to \mathrm{H}^0(X, \mathscr{O}_X) \otimes \mathrm{H}^1(D, \Theta_D) \to \mathrm{H}^1(V_0, \Theta_2) \to \mathrm{H}^1(X, \mathscr{O}_X) \otimes \mathrm{H}^0(D, \Theta_D) \to 0.$$

But we know that $\mathrm{H}^0(D, \Theta_D)$ is the Lie algebra $\mathfrak{a}$ of the group

$$A = \mathrm{GL}(2, \mathbb{C})/\mathbb{C}^* = \mathrm{SL}(2, \mathbb{C})/\{\pm 1\}$$

of automorphisms of D, and that $\mathrm{H}^1(D, \Theta_D) = 0$, as we can easily see by taking a cover of D by two open subsets. We have already seen (in [Talk no. 1]) that, if $X = E/\Gamma$, then $\mathrm{H}^1(X, \mathscr{O}) = \mathrm{Hom}(\Gamma, \mathbb{C})/\mathrm{Hom}_{\mathbb{C}}(E, \mathbb{C})$ is of dimension 2. So $\mathrm{H}^1(V_0, \Theta_2) = \mathrm{H}^1(X, \mathscr{O}) \otimes \mathfrak{a}$ is of dimension 6. The cup product

$$\mathrm{H}^1(V_0, \Theta_2) \otimes \mathrm{H}^1(V_0, \Theta_2) \to \mathrm{H}^2(V_0, \Theta_2)$$

is given by the formula

$$[(\gamma \otimes \alpha) \smile (\gamma' \otimes \alpha')] = (\gamma \smile \gamma') \otimes [\alpha, \alpha'].$$

The cone of elements $\varphi \in \mathrm{H}^1(V_0, \Theta_2)$ such that $[\varphi \smile \varphi] = 0$ can be identified with the cone of rank 1 tensors in $\mathrm{H}^1(X, \mathscr{O}) \otimes \mathfrak{a}$. Indeed, if $\varphi = \gamma \otimes \alpha$, then

$$[\varphi \smile \varphi] = (\gamma \smile \gamma) \otimes [\alpha, \alpha] = 0 \otimes 0 = 0$$

and, if φ is not a simple tensor, then we have

$$\varphi = \gamma \otimes \alpha + \gamma' \otimes \alpha'$$

with γ and γ' independent, and α and α' independent, so

$$[\varphi \smile] = 2(\gamma \smile \gamma') \otimes [\alpha, \alpha'] \neq 0.$$

2 The mixed space V

In this example, every element of $\mathrm{H}^1(V_0, \Theta_2)$ whose primary obstruction is zero is a deformation vector. More precisely: p. 2-07

Proposition 2. *There exists a mixed space $\pi \colon V \to B$ and a point $b_0 \in B$ such that*

1. *$\pi^{-1}(b_0) = V_0$ (the manifold defined in III.1) ;*
2. *there exists an isomorphism σ from a $\mathbb{C}$-analytic space B to the cone of elements $\varphi \in \mathrm{H}^1(V_0, \Theta_2)$ such that $[\varphi \smile \varphi] = 0$; and*
3. *for every subspace B' of B that has no singularities at b_0, the Spencer–Kodaira map ρ from the tangent space of B' at b_0 to $\mathrm{H}^1(V_0, \Theta)$ agrees with $\sigma: B' \to \mathrm{H}^1(V_0, \Theta_2)$.*

Let H be the analytic space of homomorphisms from Γ to $\mathfrak{a}$ whose images are contained in a vector subspace of $\mathfrak{a}$ that is 1-dimensional over $\mathbb{C}$ (i.e. (4×2) matrices of rank 1 with coefficients in $\mathbb{C}$). For every $h \in H$, $e \circ h$ is a homomorphism from Γ to A, where $e: \mathfrak{a} \to A$ denotes the exponential map, and we construct a manifold V_h that is fibred over X with fibre D as follows: V_h is the quotient of $E \times D$ by the equivalence relation defined by Γ acting via

$$\gamma \star (x, y) = (x + \gamma, ((e \circ h)(\gamma)) \cdot y).$$

These manifolds are the fibres of a mixed space $W \to H$, where W is the quotient of $H \times E \times D$ by the equivalence relation defined by Γ acting via

$$\gamma \star (h, x, y) = (h, x + y, (e \circ h(y)) \cdot y).$$

We now place the following equivalence relation on H: we have $h' \sim h$ if and only if $(h' - h)$ extends to an $\mathbb{C}$-linear map $f: E \to \mathfrak{a}$. Note that, if $h'(\Gamma)$ and $h(\Gamma)$ are contained in the same subspace L of $\mathfrak{a}$ of dimension 1 over $\mathbb{C}$ (or if $h' \sim h$), then we also have $f(E) \subset L$ (or $h \sim 0$ and $h' \sim 0$). In both cases, V_h and $V_{h'}$ are isomorphic, and we have an isomorphism $i_{h',h}: V_h \to V_{h'}$ defined by

$$i_{h',h}(x, y) = (x, e \circ f(x) \cdot y)$$

(in the first case), or

$$i_{h',h} = i_{h',0} \circ i_{0,h}$$

(in the second case). If h, h', and h'' are in the same class, then we have $i_{h''h} = i_{h''h'} \circ i_{h'h}$, *p. 4-08*
and we can place on W the equivalence relation

$$(h', z') \sim (h, z) \iff h' \sim h \text{ or } z' = i_{h'h} z$$

for $h, h' \in H$, $z \in V_h$, and $z' \in V_{h'}$.

Let B and V be the quotients of H and W (respectively) by these equivalence relations. We have a projection $V \to B$. To show that the structures of a $\mathbb{C}$-analytic space on H and W induce structures of a $\mathbb{C}$-analytic space on their quotients B and V, it suffices to remark that we can lift B to a analytic subspace of H: let, for example, $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ be a basis of Γ such that (γ_1, γ_2) is a basis of E over $\mathbb{C}$; then each class $b \in B$ contains exactly one element $h \in H$ such that

$$h(\gamma_1) = h(\gamma_2) = 0.$$

3 Calculating ρ_0

Let T be the Zariski tangent space of B at b_0, i.e. the dual of $\mathfrak{I}/\mathfrak{I}^2$, where $\mathfrak{I}$ is the ideal of germs at b_0 of analytic functions on B that are zero at b_0. Then T_0 can be identified with

$\mathrm{Hom}(\Gamma,\alpha)/\mathrm{Hom}_{\mathbb{C}}(E,\alpha)$. Also,

$$\begin{aligned}\mathrm{H}^1(V_0,\Theta) &= \mathrm{H}^1(V_0;\Theta_1)\oplus\mathrm{H}^1(V_0;\Theta_2)\\ &= \left(\mathrm{H}^1(X;\mathscr{O})\otimes E\right)\oplus\left(\mathrm{H}^1(X;\mathscr{O})\otimes\alpha\right),\end{aligned}$$

and the second term of this term can be identified with the quotient $\mathrm{Hom}(\Gamma,\alpha)/\mathrm{Hom}_{\mathbb{C}}(E,\alpha)$. We are going to show that the map $\rho_0\colon T_0\to\mathrm{H}^1(V_0;\Theta)$ is exactly the canonical injection defined by these identifications.

Let $u\in T_0=\mathrm{Hom}(\Gamma,\alpha)/\mathrm{Hom}(E,\alpha)$ be the class of an element $h\in\mathrm{Hom}(\Gamma,\alpha)$, which we suppose to be of rank 1. Then we can write h in the form $\eta\otimes\sigma$, where $\eta\in\mathrm{Hom}(\Gamma,\mathbb{C})$, $\sigma\in\alpha$, and we can consider h as a tangent vector to H at 0. Let $\overline{h}$ be the field of tangent vectors to $H\times E\times D$ at $0\times E\times D$ that projects onto h, and thus whose components over $E\times D$ are zero. Let (U_i) be a cover of $X=E/\Gamma$ by simply connected open subsets, and choose, for each *p. 4-09* i, a component $\widetilde{U}_i$ of the inverse image of U_i in E. We will denote by v_i the image over $U_i\times D$ of the field $\overline{h}|\widetilde{U}_i\times D$. This is a projectable holomorphic field on $0\times U_i\times D$ of tangent vectors of $H\times U_i\times D$, and we set $w_{ij}=v_j-v_i$, so that w_{ij} is a vertical holomorphic field on $U_{ij}\times D$, and these fields form a cocycle whose cohomology class will be, by definition, $\rho_0(u)$.

Let $x\in U_{ij}$, and let $\widetilde{x}_i$ and $\widetilde{x}_j$ be its inverse image in $\widetilde{U}_i$ and $\widetilde{U}_j$ (respectively). We have that $\widetilde{x}_j=\widetilde{x}_i+\gamma_{ij}(x)$, where $\gamma_{ij}(x)\in\Gamma$, and

$$w_{ij}(x)=\overline{h}(\widetilde{x}_j)-[\gamma_{ij}(x)]_*(\overline{h}(\widetilde{x}_i))=-h(\gamma_{ij}(x))\in\alpha.$$

Now w_{ij} is a vector field on D, and so

$$(w_{ij})\in\mathrm{Z}^1(V_0,(U_i\times D);\Theta_2),$$

and w_{ij} is of the form $\zeta\otimes\alpha$, where $\zeta\in\mathrm{Z}^1(V_0,(U_i\times D);\mathscr{O})$ is the cocycle defined by $\zeta_{ij}(x)=-\eta(\gamma_{ij}(x))$. This is a cocycle whose cohomology class is (up to a sign) the element of $\mathrm{H}^1(V_0,\mathscr{O})$ that is identified with the class η in $\mathrm{Hom}(\Gamma,\mathbb{C})/\mathrm{Hom}_{\mathbb{C}}(E,\mathbb{C})$. QED.

Appendix: Higher obstructions

I Definition of obstructions

1 The sheaf of germs of vertical automorphisms

p. 4-10

Let V_0 be a $\mathbb{C}$-analytic manifold, which we assume to be compact, and B a $\mathbb{C}$-analytic space, and let $b_0\in B$.

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