Techniques of descent and existence theorems in algebraic geometry IV. *Hilbert schemes*

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Translator's note.

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What follows is a translation of the French seminar talk:

GROTHENDIECK, A. Technique de descente et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert. *Séminaire Bourbaki*, Volume **13** (1960–61), Talk no. 221.

Contents

*[Trans.] We have made changes throughout the text following the errata (*Séminaire Bourbaki ***14***, 1961–62, Complément); we preface them with "[Comp.]".*

!TODO! additif
!TODO! errata

Introduction

The techniques described in [2, I and II] were, for the most part, independent of any projective hypotheses on the schemes in question. Unfortunately, they have not as of yet allowed us to solve the existence problems posed in [2, II]. In the current exposé, and the following, we will solve these problems by imposing projective hypotheses. The techniques used are typically projective, and practically make no use of results from [2, I and II]. Here we will construct "Hilbert schemes", which are meant to replace the use of Chow coordinates, as was mentioned in [2, II, §2]. In the next exposé, the theory of *p. 221-01*

[*] https://thosgood.com/translations

passing to the quotient in schemes, developed in [2, III], combined with the theory of Hilbert schemes, will allow us, for example, to construct Picard schemes (defined in [2, II, §3]) under rather general conditions.

In summary, we can say that we now have a more or less satisfying technique of projective constructions, apart from the fact that we are still missing[1] a theory of passing to the quotient by groups such as the projective group, acting "without fixed points" (cf. [2, III, §8]). The situation even seems slightly better in analytic geometry (if we restrict to the study of "projective" analytic spaces over a given analytic space), since, for analytic spaces, the difficulty of passing to the quotient by a group acting nicely disappears. Also, in algebraic geometry, as well as in analytic geometry, it remains to develop a technique of construction that works without any projective hypotheses.

1 Bounded sets of sheaves: invariance properties

Let k be a field, and X a k-prescheme (which we take to be of finite type, for simplicity). For every extension K/k, we obtain a K-prescheme $X_K = X \otimes_k K$. If $\mathscr{F}$ is a coherent sheaf on X_K, and if K' is an extension of K, then $\mathscr{F} \otimes_K K' = \mathscr{F}_{K'}$ is a quasi-coherent sheaf on $X_K \otimes_K K' = X_{K'}$. So, if K and K' are arbitrary extensions of k, and $\mathscr{F}$ a quasi-coherent sheaf on X_K and $\mathscr{F}'$ a quasi-coherent sheaf on $X_{K'}$, then we say that $\mathscr{F}$ and $\mathscr{F}'$ are *equivalent* if there exists an extension K''/k along with k-homomorphisms $K \to K''$ and $K' \to K''$ such that $\mathscr{F}_{K''}$ and $\mathscr{F}'_{K''}$ are isomorphic on $X_{K''}$ (**!TODO!** ? is this right?). This defines an equivalence relation, and we are interested in the equivalence classes of sheaves under this relation, and of sets of such equivalence classes. Note that, if X_0 is of finite type over k, then every class of coherent sheaves can be defined by a coherent sheaf on X_K, where K is some extension of k *of finite type*. We can thus, in the definition of classes of coherent sheaves, restrict ourselves to *algebraically closed* extensions of k, and we can also limit ourselves to a fixed algebraically closed extension Ω of k, of infinite transcendence degree; two coherent sheaves $\mathscr{F}$ and $\mathscr{F}'$ on X_Ω are then equivalent if and only if there exists a K-automorphism σ of Ω such that $\mathscr{F} \otimes_K (\Omega, \sigma)$ is isomorphic to $\mathscr{F}'$. Note that there is a bijective correspondence between classes of coherent sheaves under the first definition and under the second. p. 221-02

Let E and E' be two sets of classes of coherent sheaves on X. Consider the classes of all sheaves of the form $\mathscr{F} \otimes \mathscr{F}'$, where $\mathscr{F}$ and $\mathscr{F}'$ are coherent sheaves on the *same* X_K, with the class of $\mathscr{F}$ being in E and the class of $\mathscr{F}'$ being in E'. We thus define a set of classes of coherent sheaves that we denote by $E \otimes E'$. We can similarly define $\underline{\mathrm{Tor}}_i(E, E')$, etc.

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[2] Grothendieck, A. Technique de descente et théorèmes d'existence en géométrie algébrique, I, II, III *Séminaire Bourbaki* **12** (1959–60), Talk no. 190; **12** (1959–60), Talk no. 195; **13** (1960–61), Talk no. 212.

[1] See the addendum at the end of this exposé.

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