Regular singular differential equations

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Translator's note.

This text is one of a series[*] of translations of various papers into English. The translator takes full responsibility for any errors introduced in the passage from one language to another, and claims no rights to any of the mathematical content herein.

What follows is a translation of the French book:

DELIGNE, P. *Equations Différentielles à Points Singuliers Réguliers.* Springer–Verlag, Lecture Notes in Mathematics **163** (1970). https://publications.ias.edu/node/355

We have also made changes following the errata, which was written in April 1971, by P. Deligne, at Warwick University.

[*] https://thosgood.com/translations

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Chapter 0

Introduction

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If X is a (non-singular) complex-analytic manifold, then there is an equivalence between the notions of

a) local systems of complex vectors on X ; and

b) vector bundles on X endowed with an integrable connection.

The latter of these two notions can be adapted in an evident way to the case where X is a non-singular algebraic variety over a field k (which we will take here to be of characteristic 0). However, general algebraic vector bundles with integrable connections are pathological (see (II.6.19)); we only obtain a reasonable theory if we impose a "regularity" condition at infinity. By a theorem of Griffiths [8], this condition is automatically satisfied for "Gauss-Manin connections" (see (II.7)). In dimension one, this is closely linked to the idea of regular singular points of a differential equation (see (I.4) and (II.1)).

In Chapter I, we explain the different forms that the notion of an integrable connection can take. In Chapter II, we prove the fundamental facts concerning regular connections. In Chapter III, we translate certain results that we have obtained into the language of Nilsson class functions, and, as an application of the regularity theorem ((II.7)), we explain the proof by Brieskorn [5] of the monodromy theorem.

These notes came from the non-crystalline part of a seminar given at Harvard during the autumn of 1969, under the title: "Regular singular differential equations and crystalline cohomology".

I thank the assistants of this seminar, who had to be subjected to often unclear talks, and who allowed me to find numerous simplifications.

I also thank N. Katz, with whom I had numerous and useful conversations, and to whom are due the principal results of section (II.1).

Notation and terminology

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Within a single chapter, the references follow the decimal system. A reference to a different chapter (resp. to the current introduction) is preceded by the Roman numeral of the chapter (resp. by 0).

We will use the following definitions:

(0.1) *analytic space*: the analytic spaces are complex and of locally-finite dimension. They are assumed to be σ-compact, but not necessarily separated.

(0.2) *multiform function*: a synonym for multivalued function — for a precise definition, see (I.6.2).

(0.3) *immersion*: following the tradition of algebraic geometers, immersion is a synonym for "embedding".

(0.4) *smooth*: a morphism $f : X \to S$ of analytic spaces is smooth if, locally on X, it is isomorphic to the projection from $D^n \times S$ to S, where D^n is an open polydisc.

(0.5) *locally paracompact*: a topological space is locally paracompact if every point has a paracompact neighbourhood (and thus a fundamental system of paracompact neighbourhoods).

(0.6) non-singular (or smooth) *complex algebraic variety*: a smooth scheme of finite type over Spec($\mathbb{C}$).

(0.7) (complex) *analytic manifold*: a non-singular (or smooth) analytic space.

(0.8) *covering*: following the tradition of topologists, a covering is a continuous map $f : X \to Y$ such that every point $y \in Y$ has a neighbourhood V such that $f|V$ is isomorphic to the projection from $F \times V$ to V, where F is discrete.

Chapter I

Dictionary

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In this chapter, we explain the relations between various aspects and various uses of the notion of "local systems of complex vectors". The equivalence between the points of view considered has been well known for a long time.

We do not consider the "crystalline" point of view; see [4, 10].

I.1 Local systems and the fundamental group

Definition 1.1. Let X be a topological space. A *complex local system* on X is a sheaf of complex vectors on X that, locally on X, is isomorphic to a constant sheaf $\mathbb{C}^n$ (n$\in\mathbb{N}$).

1.2. Let X be a locally path-connected and locally simply path-connected topological space, along with a basepoint $x_0 \in X$. To avoid any ambiguity, we point out that:

a) The fundamental group $\pi_1(X, x_0)$ of X at x_0 has elements given by homotopy classes of loops based at x_0;

b) If $\alpha, \beta \in \pi_1(X, x_0)$ are represented by loops a and b, then $\alpha\beta$ is represented by the loop ab obtained by juxtaposing b and a, in that order.

Let $\mathscr{F}$ be a locally constant sheaf on X. For every path $a: [0,1] \to X$, the inverse image $a^*\mathscr{F}$ of $\mathscr{F}$ on $[0,1]$ is a locally constant, and thus constant, sheaf, and there exists exactly one isomorphism between $a^*\mathscr{F}$ and the constant sheaf defined by the set $(a^*\mathscr{F})_0 = \mathscr{F}_{a(0)}$. This isomorphism defines an isomorphism $a(\mathscr{F})$ between $(a^*\mathscr{F})_0$ and $(a^*\mathscr{F})_1$, i.e. an isomorphism

$$a(\mathscr{F})\colon \mathscr{F}_{a(0)} \to \mathscr{F}_{a(1)}.$$

This isomorphism depends only on the homotopy class of a, and satisfies $ab(\mathscr{F}) = a(\mathscr{F}) \cdot b(\mathscr{F})$. In particular, $\pi_1(X, x_0)$ acts (on the left) on the fibre $\mathscr{F}_{x_0}$ of $\mathscr{F}$ at x_0. It is well known that:

Proposition 1.3. *Under the hypotheses of (1.2), with X connected, the functor $\mathscr{F} \mapsto \mathscr{F}_{x_0}$ is an equivalence between the category of locally constant sheaves on X and the category of sets endowed with an action by the group $\pi_1(X, x_0)$.*

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Corollary 1.4. *Under the hypotheses of (1.2), with X connected, the functor $\mathscr{F} \mapsto \mathscr{F}_{x_0}$ is an equivalence between the category of complex local systems on X and the category of complex finite-dimensional representations of $\pi_1(X, x_0)$.*

1.5. Under the hypotheses of (1.2), if $a: [0,1] \to X$ is a path, and b a loop based at $a(0)$, then $aba^{-1} = a(b)$ is a path based at $a(1)$. Its homotopy class depends only on the homotopy classes of a and b. This construction defines an isomorphism between $\pi_1(X, a(0))$ and $\pi_1(X, a(1))$.

Proposition 1.6. *Under the hypotheses of (1.5), there exists, up to unique isomorphism, exactly one locally constant sheaf of groups $\Pi_1(X)$ on X (*the fundamental groupoid*), endowed, for all $x_0 \in X$, with an isomorphism*

$$\Pi_1(X)_{x_0} \simeq \pi_1(X, x_0) \tag{1.6.1}$$

and such that, for every path $a: [0,1] \to X$, the isomorphism in (1.5) between $\pi_1(X, a(0))$ and $\pi_1(X, a(1))$ can be identified, via (1.6.1), with the isomorphism in (1.2) between $\Pi_1(X)_{a(0)}$ and $\Pi_1(X)_{a(1)}$.

If X is connected, with base point x_0, then the sheaf $\Pi_1(X)$ corresponds, via the equivalence in (1.3), to the group $\pi_1(X, x_0)$ endowed with its action over itself by inner automorphisms.

Proposition 1.7. *If $\mathscr{F}$ is a locally constant sheaf on X, then there exists exactly one action (said to be* canonical*) of $\Pi_1(X)$ on $\mathscr{F}$ that, at each $x_0 \in X$, induces the action from (1.2) of $\pi_1(X, x_0)$ on $\mathscr{F}$.*

I.2 Integrable connections and local systems

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2.1. Let X be an analytic space (0.1). We define a (holomorphic) *vector bundle* on X to be a locally free sheaf of modules that is of finite type over the structure sheaf $\mathscr{O}$ of X. If $\mathscr{V}$ is a vector bundle on X, and x a point of X, then we denote by $\mathscr{V}_{(x)}$ the free $\mathscr{O}_{(x)}$-module of finite type of germs of sections of $\mathscr{V}$. If $\mathfrak{m}_x$ is the maximal ideal of $\mathscr{O}_{(x)}$, then we define the *fibre at x of the vector bundle $\mathscr{V}$* to be the **!TO-DO!** of finite rank

$$\mathscr{V}_x = \mathscr{V}_{(x)} \otimes_{\mathscr{O}_{(x)}} \mathscr{O}_{(x)}/\mathfrak{m}_x. \tag{2.1.1}$$

If $f: X \to Y$ is a morphism of analytic spaces, then the *inverse image* of a vector bundle $\mathscr{V}$ on Y is the vector bundle $f^*\mathscr{V}$ on X given by the inverse image of $\mathscr{V}$ as a coherent module: if $f^{\bullet}\mathscr{V}$ is the sheaf-theoretic inverse image of $\mathscr{V}$, then

$$f^*\mathscr{V} \simeq \mathscr{O}_X \otimes_{f^*\mathscr{O}_Y} f^{\bullet}\mathscr{V} \tag{2.1.2}$$

In particular, if $x: P \to X$ is the morphism from the point space P to X defined by a point x of X, then

$$\mathscr{V}_x \simeq x^*\mathscr{V}. \tag{2.1.3}$$

2.2. Let X be a complex-analytic manifold (0.7) and $\mathscr{V}$ a vector bundle on X. The old school would have defined a (holomorphic) connection on $\mathscr{V}$ as the data, for every pair of points (x,y) that are first order infinitesimal neighbours in X, of an isomorphism $\gamma_{y,x}\colon \mathscr{V}_x \to \mathscr{V}_y$ that depends holomorphically on (x,y) and is such that $\gamma_{x,x} = \mathrm{Id}$.

Suitably interpreted, this "definition" coincides with the currently fashionable definition (2.2.4) given below (which we not be use in the rest of the section).

It suffices to understand "point" to mean "point with values in any analytic space": p. 6

2.2.1. *A point in an analytic space X with values in an analytic space S* is a morphism from S to X.

2.2.2. If Y is a subspace of X, then the n^{th} *infinitesimal neighbourhood* of Y in X is the subspace of X defined locally by the $(n+1)$-th power of the ideal of $\mathscr{O}_X$ that defines Y.

2.2.3. Two points $x, y \in X$ with values in S are said to be *first order infinitesimal neighbours* if the map $(x,y)\colon S \to X \times X$ that they define factors through the first order infinitesimal neighbourhood of the diagonal of $X \times X$.

2.2.4. If X is a complex-analytic manifold and $\mathscr{V}$ is a vector bundle on X, then a (*holomorphic*) *connection* γ on $\mathscr{V}$ consists of the following data:

for every pair (x,y) of points of X with values in an arbitrary analytic space S, with x and y first order infinitesimal neighbours, an isomorphism $\gamma_{x,y}\colon x^*\mathscr{V} \to y^*\mathscr{V}$; this data is subject to the conditions:

(i) (functoriality) For any $f\colon T \to S$ and any first order infinitesimal neighbours $x, y\colon S \rightrightarrows X$, we have $f^*(\gamma_{y,x}) = \gamma_{yf,xf}$.

(ii) We have $\gamma_{x,x} = \mathrm{Id}$.

2.3. Let X_1 be the first-order infinitesimal neighbourhood of the diagonal X_0 of $X \times X$, and let p_1 and p_2 be the two projections of X_1 to X. By definition, the vector bundle $P^1(\mathscr{V})$ of first-order jets of sections of $\mathscr{V}$ is the bundle $(p_1)_* p_2^*\mathscr{V}$. We denote by j^1 the first-order differential operator that sends each section of $\mathscr{V}$ to its first-order jet:

$$j^1\colon \mathscr{V} \to P^1(\mathscr{V}) \simeq \mathscr{O}_{X_1} \otimes_{\mathscr{O}_X} \mathscr{V}.$$

A connection ((2.2.4)) can be understood as a homomorphism (which is automatically an isomorphism)

$$\gamma = p_1^*\mathscr{V} \to p_2^*\mathscr{V} \tag{2.3.1}$$

which induces the identity over X_0. Since

$$\mathrm{Hom}_{X_1}(p_1^*\mathscr{V}, p_2^*\mathscr{V}) \simeq \mathrm{Hom}(\mathscr{V}, (p_1)_* p_2^*\mathscr{V}),$$

a connection can also be understood as a ($\mathscr{O}$-linear) homomorphism p. 7

$$\mathrm{D}\colon \mathscr{V} \to P^1(\mathscr{V}) \tag{2.3.2}$$

such that the obvious composite arrow

$$\mathcal{V} \xrightarrow{\mathrm{D}} P^1(\mathcal{V}) \to \mathcal{V}$$

is the identity. The sections $\mathrm{D}s$ and $j^1(s)$ of $P^1(v)$ thus have the same image in $\mathcal{V}$, and $j^1(s) - \mathrm{D}(s)$ can be identified with a section ∇s of $\Omega^1_X \otimes \mathcal{V} \simeq \mathrm{Ker}(P^1(\mathcal{V}) \to \mathcal{V})$:

$$\nabla\colon \mathcal{V} \to \Omega^1(X) \tag{2.3.3}$$

$$j^1(s) = \mathrm{D}(s) + \nabla s. \tag{2.3.4}$$

In other words, a connection (2.2.4), allowing us to compare two neighbouring fibres of $\mathcal{V}$, also allows us to define the differential ∇s of a section of $\mathcal{V}$.

Conversely, equation (2.3.4) allows us to define D, and thus γ, from the covariant derivative ∇. For D to be linear, it is necessary and sufficient for ∇ to satisfy the identity

$$\nabla(fs) = \mathrm{d}f \cdot s + f \cdot \nabla s \tag{2.3.5}$$

Definition (2.2.4) is thus equivalent to the following definition, due to J.L. Koszul.

Definition 2.4. Let $\mathcal{V}$ be a (holomorphic) vector bundle on a complex-analytic manifold X. A *holomorphic connection* (or simply, *connection*) on $\mathcal{V}$ is a $\mathbb{C}$-linear homomorphism

$$\nabla\colon \mathcal{V} \to \Omega^1_X(\mathcal{V}) = \Omega^1_X \otimes_{\mathcal{O}} \mathcal{V}$$

that satisfies the Leibniz identity ((2.3.5)) for local sections f of $\mathcal{O}$ and s of $\mathcal{V}$. We call ∇ the *covariant derivative* defined by the connection.

2.5. If the vector bundle $\mathcal{V}$ is endowed with a connection Γ with covariant derivative ∇, and if w is a holomorphic vector field on X, then we set, for every local section v of $\mathcal{V}$ over an open subset U of X,

$$\nabla_w(v) = \langle \nabla v, w \rangle \in \mathcal{V}(U).$$

We call $\nabla_w\colon \mathcal{V} \to \mathcal{V}$ the *covariant derivative along the vector field w*.

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2.6. If ${}_1\Gamma$ and ${}_2\Gamma$ are connections on X, with covariant derivatives ${}_1\nabla$ and ${}_2\nabla$ (respectively), then ${}_2\nabla - {}_1\nabla$ is a $\mathcal{O}$-linear homomorphism from $\mathcal{V}$ to $\Omega^1_X(\mathcal{V})$. Conversely, the sum of ${}_1\nabla$ and such a homomorphism defines a connection on $\mathcal{V}$. Thus connections on $\mathcal{V}$ form a principal homogeneous space (or torsor) on $\underline{\mathrm{Hom}}(\mathcal{V}, \Omega^1_X(\mathcal{V})) \simeq \Omega^1_X(\underline{\mathrm{End}}(\mathcal{V}))$.

2.7. If vector bundles are endowed with connections, then every vector bundle obtained by a "tensor operation" is again endowed with a connection. This is evident with (2.2.4). More precisely, let $\mathcal{V}_1$ and $\mathcal{V}_2$ be vector bundles endowed with connections with covariant derivatives ∇_1 and ∇_2.

2.7.1. We define a connection on $\mathcal{V}_1 \oplus \mathcal{V}_2$ by the formula

$$\nabla_w(v_1 + v_2) = {}_1\nabla_w(v_1) + {}_2\nabla_w(v_2)$$

2.7.2. We define a connection on $\mathcal{V}_1 \otimes \mathcal{V}_2$ by the Leibniz formula

$$\nabla_w(v_1 \otimes v_2) = \nabla_w v_1 \cdot v_2 + v_1 \cdot \nabla_w v_2.$$

2.7.3. We define a connection on $\underline{\mathrm{Hom}}(\mathcal{V}_1, \mathcal{V}_2)$ by the formula

$$(\nabla_w f)(v_1) = {}_2\nabla_2(f(v_1)) - f({}_1\nabla v_1).$$

The canonical connection on $\mathcal{O}$ is the connection for which $\nabla f = \mathrm{d}f$.
Let $\mathcal{V}$ be a vector bundle endowed with a connection.

2.7.4. We define a connection on the dual $\mathcal{V}^\vee$ of $\mathcal{V}$ via (2.7.3) and the defining isomorphism $\mathcal{V}^\vee = \underline{\mathrm{Hom}}(\mathcal{V}, \mathcal{O})$. We have

$$\langle \nabla_w v', v\rangle = \partial_w \langle v', v\rangle - \langle v', \nabla_w v\rangle.$$

We leave it to the reader to verify that these formulas do indeed define connections. For (2.7.2), for example, one must verify that, firstly, the given formula defines a $\mathbb{C}$-bilinear map from $(\mathcal{V}_1 \otimes \mathcal{V}_2)$, which means that the right-hand side $\mathrm{II}(v_1, v_2)$ is $\mathbb{C}$-bilinear and such that $\mathrm{II}(f v_1, v_2) = \mathrm{II}(v_1, f v_2)$; secondly, one must also verify identity (2.3.5).

2.8. An $\mathcal{O}$-homomorphism f between vector bundles $\mathcal{V}_1$ and $\mathcal{V}_2$ endowed with connections is said to be *compatible with the connections* if *p. 9*

$${}_2\nabla \cdot f = f \cdot {}_1\nabla.$$

By (2.7.3), this reduces to saying that $\nabla f = 0$, if f is thought of as a section of $\underline{\mathrm{Hom}}(\mathcal{V}_1, \mathcal{V}_2)$. For example, by (2.7.3), the canonical map

$$\mathrm{Hom}(\mathcal{V}_1, \mathcal{V}_2) \otimes \mathcal{V}_1 \to \mathcal{V}_2$$

is compatible with the connections.

2.9. A local section v of $\mathcal{V}$ is said to be *horizontal* if $\nabla v = 0$. If f is a homomorphism between bundles $\mathcal{V}_1$ and $\mathcal{V}_2$ endowed with connections, then it is equivalent to say either that f is horizontal, or that f is compatible with the connections (2.8).

2.10. Let $\mathcal{V}$ be a holomorphic vector bundle on X. Define $\Omega_X^p = \bigwedge^p \Omega_X^1$ and $\Omega_X^p(\mathcal{V}) = \Omega_X^p \otimes_{\mathcal{O}} \mathcal{V}$ (the sheaf of *exterior differential p-forms with values in $\mathcal{V}$*). Suppose that $\mathcal{V}$ is endowed with a holomorphic connection. We then define $\mathbb{C}$-linear morphisms

$$\nabla\colon \Omega_X^p(\mathcal{V}) \to \Omega_X^{p+1}(\mathcal{V}) \tag{2.10.1}$$

characterised by the following formula:

$$\nabla(\alpha, v) = \mathrm{d}\alpha \cdot v + (-1)^p \alpha \wedge \nabla v, \tag{2.10.2}$$

where α is any local section of Ω^p, v is any local section of $\mathcal{V}$, and d is the exterior differential. To prove that the right-hand side $\mathrm{II}(\alpha, v)$ of (2.10.2) defines a homomorphism (2.10.1), it suffices to show that $\mathrm{II}(\alpha, v)$ is $\mathbb{C}$-bilinear and satisfies

$$\mathrm{II}(f\alpha, v) = \mathrm{II}(\alpha, fv).$$

But we have that

$$\begin{aligned}\mathrm{II}(f\alpha, v) &= \mathrm{d}(f\alpha)v + (-1)^p f\alpha \wedge \nabla v \\ &= \mathrm{d}\alpha \cdot fv + \mathrm{d}f \wedge \alpha v + (-1)^p f\alpha \wedge \nabla v \\ &= \mathrm{d}\alpha \cdot fv + (-1)^p \alpha \wedge (f\nabla v + \mathrm{d}f \cdot v) \\ &= \mathrm{II}(\alpha, fv).\end{aligned}$$

Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be vector bundles endowed with connections, and let $\mathcal{V}$ be their tensor product (2.7.2). We denote by $\wedge$ the evident maps

$$\wedge\colon \Omega^p(\mathcal{V}_1) \otimes \Omega^1(\mathcal{V}_2) \to \Omega^{p+q}(\mathcal{V})$$

such that, for any local section α (resp. β, resp. v_1, resp. v_2) of Ω^p (resp. Ω^q, resp. $\mathcal{V}_1$, resp. $\mathcal{V}_2$), we have that $(\alpha \otimes v_1) \wedge (\beta \otimes v_2) = (\alpha \wedge \beta) \otimes (v_1 \otimes v_2)$. If ν_1 (resp. ν_2) is any local section of $\Omega^p(\mathcal{V}_1)$ (resp. $\Omega^q(\mathcal{V}_2)$), then p. 10

$$\nabla(\nu_1 \wedge \nu_2) = \nu_1 \wedge \nu_2 + (-1)^p \nu_1 \wedge \nu_2. \tag{2.10.3}$$

Indeed, if $\nu_1 = \alpha v_1$ and $\nu_2 = \beta v_2$, then

$$\begin{aligned}\nabla(\nu_1 \wedge \nu_2) &= \nabla(\alpha \wedge \beta \otimes v_1 \otimes v_2) \\ &= \mathrm{d}(\alpha \wedge \beta) v_1 \otimes v_2 + (-1)^{p+q} \alpha \wedge \beta \wedge \nabla(v_1 \otimes v_2) \\ &= \mathrm{d}\alpha \wedge \beta v_1 \otimes v_2 + (-1)^p \alpha \wedge \mathrm{d}\beta v_1 \otimes v_2 \\ &\quad + (-1)^{p+q} \alpha \wedge \beta \wedge \nabla v_1 \otimes v_2 + (-1)^{p+q} \alpha \wedge \beta v_1 \wedge \nabla v_2 \\ &= \mathrm{d}\alpha v_1 \wedge \nu_2 + (-1)^p \nu_1 \wedge \mathrm{d}\beta v_2 + (-1)^p \alpha \wedge \nabla v_1 \wedge \nu_2 \\ &\quad + (-1)^{p+q} \nu_2 \wedge \beta \wedge \nabla v_2 \\ &= (\mathrm{d}\alpha v_1 + (-1)^p \alpha \wedge \nabla v_1) \wedge \nu_2 + (-1)^p \nu_1 \wedge (\mathrm{d}\beta v_2 + (-1)^q \beta \wedge \nabla v_2) \\ &= \nabla \nu_1 \wedge \nu_2 + (-1)^p \nu_1 \wedge \nabla \nu_2.\end{aligned}$$

Let $\mathcal{V}$ be a vector bundle endowed with a connection. If we apply the above formula to $\mathcal{O}$ and $\mathcal{V}$, then, for any local section α (resp. ν) of Ω^p (resp. $\Omega^q(\mathcal{V})$), we have that

$$\nabla(\alpha \wedge \nu) = \mathrm{d}\alpha \wedge \nu + (-1)^p \alpha \wedge \nabla \nu. \tag{2.10.4}$$

Iterating this formula gives

$$\begin{aligned}\nabla\nabla(\alpha \wedge \nu) &= \nabla(\mathrm{d}\alpha \wedge \nu + (-1)^p \alpha \wedge \nabla \nu) \\ &= \mathrm{d}\alpha \wedge \nu + (-1)^{p+1} \mathrm{d}\alpha \wedge \nabla \nu + (-1)^p \mathrm{d}\alpha \wedge \nabla \nu + \alpha \wedge \nabla\nabla \nu \\ &= \alpha \wedge \nabla\nabla \nu.\end{aligned} \tag{2.10.5}$$

Definition 2.11. Under the hypotheses of (2.10), the *curvature* R of the given connection on $\mathcal{V}$ is the composite homomorphism

$$R\colon \mathcal{V} \to \Omega^2_X(\mathcal{V})$$

considered as a section of $\mathrm{Hom}(\mathcal{V},\Omega^2_X(\mathcal{V})) \simeq \Omega^2_X(\mathrm{End}(\mathcal{V}))$.

2.12. Taking $q=0$ in (2.10.4) gives

$$\nabla\nabla(\alpha v) = \alpha \wedge R(v), \tag{2.12.1}$$

which we write as p. 11

$$\nabla\nabla(v) = R \wedge v \qquad \text{(the \textit{Ricci identity})}. \tag{2.12.2}$$

We endow $\underline{\mathrm{End}}(\mathcal{V})$ with the connection given in (2.7.3). The equation $\nabla(\nabla\nabla) = (\nabla\nabla)\nabla$ can be written as $\nabla(R \wedge v) = R \wedge \nabla v$. By (2.7.3), we have that $\nabla R \wedge v = \nabla(R \wedge v) - R \wedge \nabla v$, so that

$$\nabla R = 0 \qquad \text{(the \textit{Bianchi identity})}. \tag{2.12.3}$$

2.13. If α is an exterior differential p-form, then we know that

$$\begin{aligned}\langle \mathrm{d}\alpha, X_0 \wedge \ldots \wedge X_p\rangle = &\sum_i (-1)^i j_{X_i} \langle \alpha, X_0 \wedge \ldots \wedge \widehat{X_i} \wedge \ldots \wedge X_p\rangle \\ &+ \sum_{i<j})(-1)^{i+j} \langle \alpha, [X_i, X_j] \wedge X_0 \wedge \ldots \wedge \widehat{X_i} \wedge \ldots \wedge \widehat{X_j} \wedge \ldots \wedge X_p\rangle.\end{aligned}$$

From this formula, and from (2.10.2), we see that, for any local section v of $\Omega^p_X(\mathcal{V})$, and holomorphic vector fields $X_0, \ldots, X_p$,

$$\begin{aligned}\langle \nabla v, X_0 \wedge \ldots \wedge X_p\rangle = &\sum_i (-1)^i \nabla_{X_i} \langle v, X_0 \wedge \ldots \wedge \widehat{X_i} \wedge \ldots \wedge X_p\rangle \\ &+ \sum_{i<j})(-1)^{i+j} \langle v, [X_i, X_j] \wedge X_0 \wedge \ldots \wedge \widehat{X_i} \wedge \ldots \wedge \widehat{X_j} \wedge \ldots \wedge X_p\rangle.\end{aligned} \tag{2.13.1}$$

In particular, for any local section v of $\mathcal{V}$, we have that

$$\langle \nabla\nabla v, X_1 \wedge X_2\rangle = \nabla_{X_1}\langle \nabla v, X_2\rangle - \nabla_{X_2}\langle v, X_1\rangle - \langle \nabla v, [X_1, X_2]\rangle.$$

That is,

$$R(X_1, X_2)(v) = \nabla_{X_1}\nabla_{X_2} v - \nabla_{X_2}\nabla_{X_1} v - \nabla_{[X_1,X_2]} v. \tag{2.13.2}$$

Definition 2.14. A connection is said to be *integrable* if its curvature is zero, i.e. (2.13.2) if the following holds identically:

$$\nabla_{[X,Y]} = [\nabla_X, \nabla_Y].$$

If $\dim(X) \leqslant 1$, then every connection is integrable.

If Γ is an integrable connection on $\mathcal{V}$, then the morphism ∇ of (2.10.1) satisfy $\nabla\nabla = 0$, and so the $\Omega^p(\mathcal{V})$ give a differential complex $\Omega^\bullet(\mathcal{V})$.

Definition 2.15. Under the above hypotheses, the complex $\Omega^\bullet(\mathcal{V})$ is called the *holomorphic de Rham complex* with values in $\mathcal{V}$. p. 12

The results (2.16) to (2.19) that follow will be proven in a more general setting in (2.23).

Proposition 2.16. *Let V be a local complex system on a complex-analytic variety X (0.6), and let $\mathcal{V} = \mathcal{O} \otimes_{\mathbb{C}} V$.*

(i) *There exists, on the vector bundle $\mathcal{V}$, exactly one connection (said to be* canonical*) whose horizontal sections are the local sections of the subsheaf V of $\mathcal{V}$.*

(ii) *The canonical connection on $\mathcal{V}$ is integrable.*

(iii) *For any local section f (resp. v) of $\mathcal{O}$ (resp. V),*

$$\nabla(fv) = \mathrm{d}f \cdot v. \tag{2.16.1}$$

Proof. If ∇ satisfies (i), then (2.16.1) is a particular case of (2.3.5). Conversely, the right-hand side $\mathrm{II}(f,v)$ of (2.16.1) is $\mathbb{C}$-bilinear, and thus extends uniquely to a $\mathbb{C}$-linear map $\nabla\colon \mathcal{V} \to \Omega^1(\mathcal{V})$, which we can show defines a connection. Claim (ii) is local on X, which allows us to reduce to the case where $V = \underline{\mathbb{C}}$. Then $\mathcal{V} = \mathcal{O}$, $\nabla = \mathrm{d}$, and $\nabla_{[X,Y]} = [\nabla_X, \nabla_Y]$ by the definition of $[X,Y]$. □

It is well known that:

Theorem 2.17. *Let X be a complex-analytic variety. Then the following functors are quasi-inverse to one another, and thus give an equivalence between the category of complex local systems on X and the category of holomorphic vector bundles with on X with integrable connections (with the morphisms being the horizontal morphisms of vector bundles):*

a) *the complex local system V is sent to $\mathcal{V} = \mathcal{O} \otimes V$ endowed with its canonical connection;*

b) *the holomorphic vector bundle $\mathcal{V}$ endowed with its integrable connection is sent to the subsheaf V of $\mathcal{V}$ consisting of horizontal sections (i.e. those v such that $\nabla v = 0$).*

These equivalences are compatible with taking the tensor product, the internal Hom, and the dual; to the unit complex local system $\underline{\mathbb{C}}$ corresponds the bundle $\mathcal{O}$ endowed with the connection ∇ such that $\nabla f = \mathrm{d}f$.

Definition (2.10.2) implies the following:

Proposition 2.18. *If V is a complex local system on X, and if $\mathcal{V} = \mathcal{O} \otimes_{\mathbb{C}} V$, then the system of isomorphisms* p. 13

$$\Omega_X^p \otimes_{\mathbb{C}} V \simeq \Omega_X^p \otimes_{\mathcal{O}} \mathcal{O} \otimes_{\mathbb{C}} V \simeq \Omega_X^p \otimes_{\mathcal{O}} \mathcal{V}$$

is an isomorphism of complexes

$$\Omega_X^\bullet \otimes_{\mathbb{C}} V \to \Omega_X^\bullet(\mathcal{V}).$$

From this, the holomorphic Poincaré lemma gives the following:

Proposition 2.19. *Under the hypotheses of (2.16), the complex $\Omega_X^\bullet(\mathcal{V})$ is a resolution of the sheaf $\mathcal{V}$.*

2.20. Variants.

2.20.1. If X is a differentiable manifold, and we consider C^∞ connections on C^∞ vector bundles, then all of the above results still hold true, mutatis mutandis. We will not use this fact.

2.20.2. Theorem (2.17) makes essential use of the non-singularity of X; it is thus unimportant to note that this hypothesis has not been used in an essential way before (2.17)

2.20.3. The definition (2.4) of a connection and the definition (2.11) of an integrable connection are formal enough that we can transport them to the category of schemes, or in relative settings:

Definition 2.21.

(i) Let $f: X \to S$ be a smooth morphism of schemes, and $\mathscr{V}$ a quasi-coherent sheaf on X. A *relative connection* on $\mathscr{V}$ is an $f^*\mathscr{O}_S$-linear sheaf morphism

$$\nabla: \mathscr{V} \to \Omega^1_{X/S}(\mathscr{V})$$

(called the *covariant derivative* defined by the connection) that identically satisfies, for any local section f (resp. v) of $\mathscr{O}_X$ (resp. $\mathscr{V}$),

$$\nabla(fv) = f \cdot \nabla v + \mathrm{d}f \cdot v.$$

(ii) Given $\mathscr{V}$ endowed with a relative connection, there exists exactly one system of $f^*\mathscr{O}_S$-homomorphisms of sheaves

$$\nabla^{(p)}: \Omega^p_{X/S}(\mathscr{V}) \to \Omega^{p+1}_{X/S}(\mathscr{V})$$

that satisfies (2.10.4) and is such that $\nabla^{(0)} = \nabla$. p. 14

(iii) The *curvature* of a connection is defined by

$$R = \nabla^{(1)}\nabla^{(0)} \in \underline{\mathrm{Hom}}(\mathscr{V}, \Omega^2_{X/S}(\mathscr{V})) \cong \Omega^2_{X/S}(\underline{\mathrm{End}}(\mathscr{V})).$$

The curvature satisfies the Ricci identity (2.12.2) and the Bianchi identity (2.12.3).

(iv) An *integrable connection* is a connection with zero curvature.

(v) The *de Rham complex* defined by an integrable connection is the complex $(\Omega^p_{X/S}(\mathscr{V}), \nabla)$.

2.22. Let $f: X \to S$ be a *smooth* morphism of complex-analytic spaces; by hypothesis, f is thus locally (in the domain) isomorphic to a projection $\mathrm{pr}_2: \mathbb{C}^n \times S \to S$ (for some $n \in \mathbb{N}$). A *local relative system* on X is a sheaf of $f^*\mathscr{O}_S$-modules that is locally isomorphic to the sheaf-theoretic inverse image of a coherent analytic sheaf on S. If $\mathscr{V}$ is a coherent analytic sheaf on X, then a *relative connection* on $\mathscr{V}$ is an $f^*\mathscr{O}_S$-linear homomorphism

$$\nabla: \mathscr{V} \to \Omega^1_{X/S}(\mathscr{V})$$

that identically satisfies, for any local section f (resp. v) of $\mathscr{O}$ (resp. $\mathscr{V}$),

$$\nabla(fv) = f \cdot \nabla v + \mathrm{d}f \cdot v.$$

A *morphism* between vector bundles endowed with relative connections is a morphism of vector bundles that commutes with ∇. We define, as in (2.11) and (2.21), the *curvature*

$R \in \Omega^2_{X/S}(\underline{\mathrm{End}}(\mathcal{V}))$ of a relative connection. A relative connection is said to be *integrable* if $R = 0$, in which case we have the *relative de Rham complex with values in* $\mathcal{V}$, denoted by $\Omega^\bullet_{X/S}(\mathcal{V})$, and defined as in (2.15) and (2.21).

The "absolute" statements (2.17), (2.18), and (2.19) have "relative" (i.e. "with parameters") analogues:

Theorem 2.23. *Under the hypotheses of (2.22), we have the following.*

(i) For every relative local system V on X, there exists a coherent analytic sheaf $\mathcal{V} = \mathcal{O}_X \otimes_{f^*\mathcal{O}_S} V$, and exactly one relative connection, said to be canonical, such that a local section v of $\mathcal{V}$ is horizontal (i.e. such that $\nabla v = 0$) if and only if v is a section of V; this connection is integrable.

(ii) Given a relative local system V on X, the de Rham complex defined by $\mathcal{V} = \mathcal{O}_X \otimes_{f^*\mathcal{O}_S} V$, endowed with its canonical connection, is a resolution of the sheaf V. p. 15

(iii) The following functors are quasi-inverse to one another, and thus give an equivalence between the category of relative local systems on X and the category of coherent analytic sheaves on X endowed with a relative integrable connection:

a) the relative local system V is sent to $\mathcal{V} = \mathcal{O}_X \otimes_{f^*\mathcal{O}_S} V$ endowed with its canonical connection;

b) the coherent analytic sheaf $\mathcal{V}$ on X endowed with a relative integrable connection is sent to the subsheaf consisting of its horizontal sections (i.e. the sections v such that $\nabla v = 0$).

Proof. We first prove (i). To show that $\mathcal{V}$ is coherent, it suffices to do so locally, for $V = f^\bullet V_0$, in which case $\mathcal{V}$ is the inverse image, in the sense of coherent analytic sheaves, of $\mathcal{V}_0$. The canonical relative connection necessarily satisfies, for any local section f (resp v_0) of $\mathcal{O}_X$ (resp. V),

$$\nabla(f v_0) = \mathrm{d}f \cdot v_0. \tag{2.23.1}$$

The right-hand side $\mathrm{II}(f, v_0)$ of this equation is biadditive in f and v_0, and satisfies, for any local section g of $f^*\mathcal{O}_S$, the identity

$$\mathrm{II}(fg, v_0) = \mathrm{II}(f, gv_0)$$

(using the fact that $\mathrm{d}g = 0$ in $\Omega^1_{X/S}$). We thus deduce the existence and uniqueness of a relative connection ∇ that satisfies (2.23.1). Finally, we have that

$$\nabla\nabla(f v_0) = \nabla(\mathrm{d}f \cdot v_0) = \mathrm{dd}f \cdot v_0 = 0,$$

and so the canonical connection ∇ is integrable. The fact that only the sections of V are horizontal is a particular case of (ii), which is proven below. □

2.23.2. We first of all consider the particular case of (ii) where $S = D^n$, $X = D^n \times D^m$, $f = \mathrm{pr}_2$, and the relative local system V is the inverse image of $\mathcal{O}_S$. The complex of global sections

$$0 \to \Gamma(f^\bullet\mathcal{O}_S) \to \Gamma(\mathcal{O}_X) \to \Gamma(\Omega^1_{X/S}) \to \ldots$$

is acyclic, since it admits the homotopy operator H defined below. p. 16

a) $H\colon \Gamma(\mathscr{O}_X) \to \Gamma(f^{\bullet}\mathscr{O}_S) = \Gamma(S,\mathscr{O}_S)$ is the inverse image under the zero section of f;

b) an element $\omega \in \Gamma(\Omega^p_{X/S})$ (where $p > 0$) can be represented in a unique way as a sum of convergent series:

$$\omega = \sum_{\substack{I\subset[1,m]\\ |I|=p}} \sum_{\underline{n}\in\mathbb{N}^{m+n}} a^I_{\underline{n}} \left(\prod_{i\in I} x_i^{n_i}\,\mathrm{d}x_i\right)\left(\prod_{i\in[1,m+n]\setminus I} x_i^{n_i}\right)$$

and we set

$$H(\omega) = \sum_{I\subset[1,m]} \sum_{j\in I} \sum_{\underline{n}\in\mathbb{N}^{m+n}} a^I_{\underline{n}} \left(\prod_{\substack{j\in I\\ i\neq j}} x_j^{n_j}\,\mathrm{d}x_j \frac{x_j^{n_j+1}}{n_j+1}\right)\left(\prod_{i\in[1,m+n]\setminus I} x_i^{n_i}.\right)$$

This remains true if we replace D^{m+n} by a smaller polycylinder, and so $\Omega^{\bullet}_{X/S}$ is a resolution of $f^{\bullet}\mathscr{O}_S$.

2.23.3. We now prove (ii), which is of a local nature on X and S. Denoting by D the open unit disc, we can thus restrict to the case where S is a closed analytic subset of the polycylinder D^n, where $X = D^m \times S$, with $f = \mathrm{pr}_2$, and where V is the inverse image of a coherent analytic sheaf V_0 on S. Applying the syzygy theorem, and possibly shrinking X and S, we can further suppose that the direct image of V_0 on D^n, which we again denote by V_0, admits a finite resolution $\mathscr{L}$ by free coherent $\mathscr{O}_{D^n}$-modules. To prove (ii), we are allowed to replace V_0 by its direct image on D^n, and to suppose that $D^n = S$, which we now do.

If Σ_0 is a short exact sequence of coherent $\mathscr{O}_S$-modules

$$\Sigma_0\colon 0 \to V_0' \to V_0 \to V_0'' \to 0,$$

then let $\Sigma = f^{\bullet}\Sigma_0$ be the exact sequence of relative local systems given by the inverse image of Σ_0 (which is exact since $f^{\bullet}$ is an exact functor), and let $\Omega^{\bullet}_{X/S}(\Sigma)$ be the corresponding exact sequence of relative de Rham complexes:

$$\Omega^{\bullet}_{X/S}(\Sigma)\colon 0 \to \Omega^p_{X/S} \otimes_{f^{\bullet}\mathscr{O}_S} f^{\bullet}V_0' \to \Omega^p_{X/S} \otimes_{f^{\bullet}\mathscr{O}_S} f^{\bullet}V_0 \to \Omega^p_{X/S} \otimes_{f^{\bullet}\mathscr{O}_S} f^{\bullet}V_0'' \to 0.$$

This sequence is exact since $\Omega^p_{X/S}$ is flat over $f^{\bullet}\mathscr{O}_S$, since it is locally free over $\mathscr{O}_X$ which is itself flat over $f^{\bullet}\mathscr{O}_S$.

The snake lemma applied to $\Omega^{\bullet}_{X/S}(\Sigma)$ shows that, if claim (ii) is satisfied for any two p. 17
of relative local systems $f^{\bullet}V_0$, $f^{\bullet}V_0'$, and $f^{\bullet}V_0''$, then it is again satisfied for the third. We thus deduce, by induction, that, if V_0 admits a finite resolution $M_{\bullet}$ by modules that satisfy (ii), then V_0 satisfies (ii). This, applied to V_0 and $\mathscr{L}^{\bullet}$, finishes the proof of (i) and (ii).

It follows from (ii) that the composite (iii)b∘(iii)a of the functors from (iii) is canonically isomorphic to the identity; furthermore, if V_1 and V_2 are relative local systems, and if $u\colon \mathscr{V}_1 \to \mathscr{V}_2$ is a homomorphism that induces 0 on V_1, then $u = 0$, since V_1 generates $\mathscr{V}_1$; it thus follows that the functor (iii)a is fully faithful. It remains to show that every vector bundle $\mathscr{V}$ endowed with a relative connection ∇ is given locally by a relative local system.

Case 1: *$S = D^n$, $X = D^{n+1} = D^n \times D$, $f = \mathrm{pr}_1$, and $\mathscr{V}$ is free.*

Under these hypotheses, if v is an arbitrary section of the inverse image of $\mathscr{V}$ under the zero section s_0 of f, then there exists exactly one horizontal section $\widetilde{v}$ of $\mathscr{V}$ that agrees with v on $s_0(S)$ (as follows from the existence and uniqueness of solutions for Cauchy problems with parameters). If $(e_i)_{1\leqslant i\leqslant k}$ is a basis of $s_0^*\mathscr{V}$, then the $\widetilde{e_i}$ form a horizontal basis of $\mathscr{V}$, and $(\mathscr{V},\nabla)$ is defined by the relative local system $f^\bullet s_0^*\mathscr{V} \simeq f^\bullet\mathscr{O}_S^k$.

Case 2: *$S = D^n$, $X = D^{n+1} = D^n \times D$, and $f = \mathrm{pr}_1$.*

By possibly shrinking X and S, we can suppose that $\mathscr{V}$ admits a free presentation:

$$\mathscr{V}_1 \xrightarrow{d} \mathscr{V}_0 \xrightarrow{\varepsilon} \mathscr{V} \to 0.$$

By then possibly shrinking again, we can further suppose that $\mathscr{V}_0$ and $\mathscr{V}_1$ admit connections ∇_1 and ∇_0 (respectively) such that ε and d are compatible with the connections (if (e_i) is a basis of $\mathscr{V}_0$, then ∇_0 is determined by the $\nabla_0 e_i$, and it suffices to choose $\nabla_0 e_i$ such that $\varepsilon(\nabla_0 e_i) = \nabla(\varepsilon(e_i))$; we proceed similarly for ∇_1). The connections ∇_0 and ∇_1 are automatically integrable, since f is of relative dimension 1. There thus exist (by Case 1) relative local systems V_0 and V_1 such that $(\mathscr{V}_i,\nabla_i) \simeq \mathscr{O}_X \otimes_{f^\bullet\mathscr{O}_S} V_i$. We then have that

$$(\mathscr{V},\nabla) \simeq \mathscr{O}_X \otimes_{f^\bullet\mathscr{O}_S} (V_0/dV_1).$$

p. 18

Case 3: *f is of relative dimension* 1.

We can suppose that S is a closed analytic subset of D^n, and that $X = S \times D$ and $f = \mathrm{pr}_1$. The relative local systems (resp. the modules with relative connections) on X can then be identified with the local relative systems (resp. the modules with relative connections) on $D^n \times D$ that are annihilated by the inverse image of the ideal that defines S, and we conclude by using Case 2.

General case. We proceed by induction on the relative dimension n of f. The case $n = 0$ is trivial. If $n \neq 0$, then we are led to the case where $X = S \times D^{n-1} \times D$ and $f = \mathrm{pr}_1$. The bundle with connection $(\mathscr{V},\nabla)$ induces a bundle $\mathscr{V}_0$ with connection on $X_0 = S \times D^{n-1} \times \{0\}$ which is, by induction, of the form $(\mathscr{V}_0,\nabla_0) = \mathscr{O}_{X_0} \otimes_{\mathrm{pr}_1^\bullet\mathscr{O}_S} V$. The projection f from X to $S \times D^{n-1}$ is of relative dimension 1, and the relative connection ∇ induces a relative connection for $\mathscr{V}$ on $X/S \times D^{n-1}$. By Case 3, there exists a vector bundle V_1 on $S \times D^{n-1}$, as well as an isomorphism

$$\mathscr{V} \simeq \mathscr{O}_X \otimes_{p^\bullet\mathscr{O}_{S\times D^{n-1}}} p^\bullet V_1.$$

of bundles with relative connections (with respect to p).

The vector bundle V_1 can be identified with the restriction of $\mathscr{V}$ to X_0, whence we obtain an isomorphism

$$\alpha\colon \mathscr{V} \simeq \mathscr{O}_X \otimes_{f^\bullet\mathscr{O}_S} V$$

of vector bundles, such that

(i) the restriction of α to X_0 is horizontal ; and

(ii) α is "relatively horizontal" with respect to p.

If v is a section of V, then condition (ii) implies that

$$\nabla_{x_n} v = 0.$$

If $1 \leqslant i < n$, since $R = 0$, then, by an analogous statement to (2.13.2), we have that

$$\nabla_{x_n} \nabla_{x_i} v = \nabla_{x_i} \nabla_{x_n} v = 0.$$

In other words, $\nabla_{x_i} v$ is a relative horizontal section, with respect to p, of $\mathcal{V}$; by (i), it is zero on X_0, and is thus zero, and we conclude that $\nabla v = 0$. The isomorphism α is thus horizontal, and this finishes the proof of (2.23). *p. 19*

□

Some results in general topology ((2.24) to (2.27)) will be necessary to deduce (2.28) from (2.23).

Reminder 2.24. Let Y be a closed subset of a topological space X, and suppose that Y has a paracompact neighbourhood. For every sheaf $\mathscr{F}$ on X, we have that

$$\varinjlim_{U \supset Y} \mathrm{H}^\bullet(U, \mathscr{F}) \xrightarrow{\sim} \mathrm{H}^\bullet(Y, \mathscr{F}).$$

Proof. See Godement [7, II, 4.11.1, p. 193]. □

Corollary 2.25. *Let $f : X \to S$ be a proper separated morphism between topological spaces. Suppose that S is locally paracompact (0.5). Then, for every $s \in S$, and for every sheaf $\mathscr{F}$ on X, we have that*

$$(\mathbf{R}^i f_* \mathscr{F})_s \simeq \mathrm{H}^i(f^{-1}(s), \mathscr{F} | f^{-1}(s)).$$

Proof. Since f is closed, the $f^{-1}(U)$ form a fundamental system of neighbourhoods of $f^{-1}(s)$, where the U are neighbourhoods of s. Furthermore, if U is paracompact, then $f^{-1}(U)$ is paracompact, since f is proper and separated. We conclude by (2.24). □

Reminder 2.26. Let X be a contractible locally paracompact topological space, i an integer, and V a complex local system on X, such that $\dim_{\mathbb{C}} \mathrm{H}^i(X, V) < \infty$. Then, for every vector space A over $\mathbb{C}$, possibly of infinite dimension, we have that

$$A \otimes_{\mathbb{C}} \mathrm{H}^i(X, V) \xrightarrow{\sim} \mathrm{H}^i(X, A \otimes_{\mathbb{C}} V). \tag{2.26.1}$$

Proof. We denote by $\mathrm{H}_\bullet(X,V^*)$ the singular homology of X with coefficients in V^*. The universal coefficient formula, which holds here, gives

$$\mathrm{H}^i(X,A\otimes V)\simeq \mathrm{Hom}_{\mathbb{C}}(\mathrm{H}_i(X,V^*),A). \tag{2.26.2}$$

For $A=\mathbb{C}$, we thus conclude that $\dim \mathrm{H}_i(X,V^*)<\infty$. Equation (2.26.1) then follows from (2.26.2). □

2.27. Let $f:X\to S$ be a smooth morphism of complex-analytic spaces, and let V be a local system on X. Then the sheaf p. 20

$$V_{\mathrm{rel}}=f^{\boldsymbol{\cdot}}\mathcal{O}_S\otimes_{\mathbb{C}}V \tag{2.27.1}$$

is a relative local system. We denote by $\Omega^\bullet_{X/S}(V)$ the corresponding de Rham complex. By (2.23), $\Omega^\bullet_{X/S}$ is a resolution of V_{rel}. We thus have that

$$\mathbf{R}^if_*V_{\mathrm{rel}}\xrightarrow{\sim}\mathbf{R}^if_*(\Omega^\bullet_{X/S}(V)) \tag{2.27.2}$$

where the right-hand side is the relative hypercohomology. From (2.27.1), we thus obtain an arrow

$$\mathcal{O}_S\otimes_{\mathbb{C}}\mathbf{R}^if_*V\to\mathbf{R}^if_*(V_{\mathrm{rel}}), \tag{2.27.3}$$

whence, by composition, an arrow

$$\mathcal{O}_S\otimes\mathbf{R}^if_*V\to\mathbf{R}^if_*(\Omega_{X/S}(V)). \tag{2.27.4}$$

Proposition 2.28. *Let $f:X\to S$ be a smooth separated morphism of analytic spaces, i an integer, and V a complex local system on X. We suppose that*

a) f is topologically trivial locally on S ; and

b) the fibres of f satisfy

$$\dim\mathrm{H}^i(f^{-1}(s),V)<\infty.$$

Then the arrow (2.27.4) is an isomorphism:

$$\mathcal{O}_S\otimes_{\mathbb{C}}\mathbf{R}^if_*V\xrightarrow{\sim}\mathbf{R}^if_*(\Omega^\bullet_{X/S}(V)).$$

Proof. Let $s\in S$, $Y=f^{-1}(s)$, and $V_0=V|Y$. To show that (2.27.4) is an isomorphism, it suffices to construct a fundamental system T of neighbourhoods of s such that the arrows

$$\mathrm{H}^0(T,\mathcal{O}_S)\otimes\mathrm{H}^i(T\times Y,\mathrm{pr}_2^{\boldsymbol{\cdot}}V_0)\xrightarrow{\sim}\mathrm{H}^i(T\times Y,\mathrm{pr}_1^{\boldsymbol{\cdot}}\mathcal{O}_S\otimes\mathrm{pr}_2^{\boldsymbol{\cdot}}V_0) \tag{2.28.1}$$

are isomorphisms. In fact, the fibre at s of (2.27.3), which is the inductive limit of the arrows (2.28.1), will then be an isomorphism.

We will prove (2.28.1) for a compact Stein neighbourhood T of s, assumed to be contractible. The arrow in (2.28.1) can then be written as p. 21

$$\mathrm{H}^0(T,\mathcal{O}_S)\otimes\mathrm{H}^i(Y,V_0)\xrightarrow{\sim}\mathrm{H}^i(T\times Y,\mathrm{pr}_1^{\boldsymbol{\cdot}}\mathcal{O}_S\otimes\mathrm{pr}_2^{\boldsymbol{\cdot}}V_0). \tag{2.28.2}$$

We can calculate the right-hand side of (2.28.2) by using the Leray spectral sequence for $\mathrm{pr}_2: T\times Y\to Y$. By (2.25), since $\mathrm{H}^i(T,\mathcal{O}_S)=0$, we have that

$$\mathrm{H}^i(T\times Y,\mathrm{pr}_1^{\boldsymbol{\cdot}}\mathcal{O}_S\otimes\mathrm{pr}_2^{\boldsymbol{\cdot}}V_0)\simeq\mathrm{H}^i(Y,\mathrm{H}^0(T,\mathcal{O}_S)\otimes V_0),$$

and we conclude by (2.26). □

2.29. Under the hypotheses of (2.28), with S smooth, we define the *Gauss–Manin connection* on $\mathbf{R}^i f_* \Omega^\bullet_{X/S}(V)$ as being the unique integrable connection that admits the local sections of $\mathbf{R}^i f_* V$ as its horizontal sections (2.17).

I.3 Translation in terms of first-order partial differential equations

3.1. Let X be a complex-analytic variety. If $\mathcal{V}$ is the holomorphic vector bundle defined by a $\mathbb{C}$-vector space V_0, then we have seen that $\mathcal{V}$ admits a canonical connection with covariant derivative ${}_0\nabla$. If ∇ is the covariant derivative defined by another connection on $\mathcal{V}$, then we have seen (2.6) that ∇ can be written in the form

$$\nabla = {}_0\nabla + \Gamma, \quad \text{where } \Gamma \in \Omega(\underline{\mathrm{End}}(\mathcal{V})).$$

If we identify sections of $\mathcal{V}$ with holomorphic maps from X to V_0, then we have that

$$\nabla v = \mathrm{d}v + \Gamma \cdot v \tag{3.1.1}$$

If we suppose that we have chosen a basis of V, i.e. an isomorphism $e\colon \mathbb{C}^n \to V_0$ with coordinates (identified with basis vectors) $e_\alpha\colon \mathbb{C} \to V_0$, then Γ can be written as a matrix ω^α_β of differential forms (the *matrix of forms of the connection*), and (3.1.1) can then be written as

$$(\nabla v)^\alpha = \mathrm{d}v^\alpha + \sum_\beta \omega^\alpha_\beta v^\beta. \tag{3.1.2}$$

Let $\mathcal{V}$ be an arbitrary holomorphic vector bundle on X. The choice of a basis $e\colon \mathbb{C}^n \xrightarrow{\sim} \mathcal{V}$ of $\mathcal{V}$ allows us to think of $\mathcal{V}$ as being defined by a constant vector **!TO-DO!** ($\mathbb{C}^n$), and the above remarks apply: connections on $\mathcal{V}$ correspond, via (3.1.2), with $(n \times n)$-matrices of differential forms on X. If ω_e is the matrix of the connection ∇ in the basis e, and if $f\colon \mathbb{C}^n \xrightarrow{\sim} \mathcal{V}$ is another basis of $\mathcal{V}$, with matrix $A \in \mathrm{GL}_n(\mathcal{O})$ (where $A = e^{-1}f$), then (3.1.2) p. 22

$$\begin{aligned}\nabla v &= e\mathrm{d}(e^{-1}v) + e\omega_e e^{-1} v \\ &= fA^{-1}\mathrm{d}(Af^{-1}v) + fA^{-1}\omega_e A f^{-1} v \\ &= f\mathrm{d}f^{-1}v + f(A^{-1}\mathrm{d}A + A^{-1}\omega_e A)f^{-1}v.\end{aligned}$$

Comparing this with (3.1.2) in the basis f, we find that

$$\omega_f = A^{-1}\mathrm{d}A + A^{-1}\omega_e A. \tag{3.1.3}$$

If, further, (x^i) is a system of local coordinates on X, which defines a basis $(\mathrm{d}x^i)$ of Ω^1_X, we set

$$\omega^\alpha_\beta = \sum_i \Gamma^\alpha_{\beta i} \mathrm{d}x^i$$

and we call the holomorphic functions $\Gamma^\alpha_{\beta i}$ the *coefficients of the connection*. Equation (3.1.2) can the be written as

$$(\nabla_i v)^\alpha = \partial_i v^\alpha + \sum_\beta \Gamma^\alpha_{\beta i} v^\beta. \tag{3.1.4}$$

The differential equation $\nabla v = 0$ of horizontal sections of $\mathscr{V}$ can be written as the linear homogeneous system of first-order partial differential equations

$$\partial_i v^\alpha = -\sum_\beta \Gamma^\alpha_{\beta i} v^\beta. \tag{3.1.5}$$

3.2. With the notation of (3.1.2), and using Einstein summation notation, we have that

$$\begin{aligned}
\nabla\nabla v &= \nabla((\mathrm{d}v^\alpha + \omega^\alpha_\beta v^\beta)e_\alpha) \\
&= \mathrm{d}(\mathrm{d}v^\alpha + \omega^\alpha_\beta v^\beta)e_\alpha - (\mathrm{d}v^\alpha + \omega^\alpha_\beta v^\beta) \wedge \omega^\gamma_\alpha \cdot e_\gamma \\
&= \mathrm{d}\omega^\alpha_\beta \cdot v^\beta \cdot e_\alpha - \omega^\alpha_\beta \wedge \mathrm{d}v^\beta \cdot e_\alpha - \mathrm{d}v^\alpha \wedge \omega^\gamma_\alpha \cdot e_\gamma - \omega^\alpha_\beta \wedge \omega^\gamma_\alpha \cdot v^\beta e_\gamma \\
&= (\mathrm{d}\omega^\gamma_\beta - \omega^\alpha_\beta \wedge \omega^\gamma_\alpha) v^\beta e_\gamma.
\end{aligned}$$

The curvature tensor matrix is thus p. 23

$$R^\alpha_\beta = \mathrm{d}\omega^\alpha_\beta + \sum_\gamma \omega^\alpha_\gamma \wedge \omega^\gamma_\beta, \tag{3.2.1}$$

which we can also write as

$$R = \mathrm{d}\omega + \omega \wedge \omega. \tag{3.2.2}$$

Equation (3.2.1) gives, in a system (x^i) of local coordinates,

$$\begin{cases} R^\alpha_{\beta ij} &= (\partial_i \Gamma^\alpha_{\beta j} - \partial_j \Gamma^\alpha_{\beta i}) + (\Gamma^\alpha_{\gamma i}\Gamma^\gamma_{\beta j} - \Gamma^\gamma_{\gamma j}\Gamma^\gamma_{\beta i}) \\ R^\alpha_\beta &= \sum_{i<j} R^\alpha_{\beta ij} \mathrm{d}x^i \wedge \mathrm{d}x^j. \end{cases} \tag{3.2.3}$$

The condition $R^\alpha_{\beta ij} = 0$ is the integrability condition of the system (3.1.5), in the classical sense of the word; it can be obtained by eliminating v^α from the equations given by substituting (3.1.5) into the identity $\partial_i \partial_j v^\alpha = \partial_j \partial_i v^\alpha$.

I.4 n^{th}-order differential equations

4.1. The solution of a linear homogeneous n^{th}-order differential equation

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} y = \sum_{i=1}^n a_i(x) \frac{\mathrm{d}^{n-i}}{\mathrm{d}x^i} y \tag{4.1.1}$$

is equivalent to that of the system

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}x} y_i = y_{i+1} & \text{(for } 1 \leqslant i < n\text{);} \\ \frac{\mathrm{d}}{\mathrm{d}x} y_n = \sum_{i=1}^n a_i(x) y_{n+1-i} \end{cases} \tag{4.1.2}$$

of n first-order equations.

By (3), this system can be described as the differential equation of horizontal sections of a rank-n vector bundle endowed with a suitable connection, and this is what we aim to further explore.

4.2. Let X be a non-singular complex-analytic variety of pure dimension 1. Let X_n be the n^{th} infinitesimal neighbourhood of the diagonal of $X \times X$, and p_1 and p_2 the two projections from X_n to X. We denote by $\pi_{k,l}$ the injection from X_l to X_k, for $l \leqslant k$. *p. 24*

Let $\Omega^{\otimes n}$ be the n^{th} tensor power of the invertible sheaf Ω^1_X (for $n \in \mathbb{Z}$). If I is the ideal that defines the diagonal of $X \times X$, then $I/I^2 \simeq \Omega^1_X$ canonically, and

$$I^n/I^{n+1} \simeq \Omega^{\otimes n}. \tag{4.2.1}$$

If $\mathscr{L}$ is an invertible sheaf on X, then we denote by $P^n(\mathscr{L})$ the vector bundle

$$P^n(\mathscr{L}) = (p_1)_* p_2^* \mathscr{L} \tag{4.2.2}$$

of n^{th}-order jets of sections of $\mathscr{L}$. The I-adic filtration of $p_2^*\mathscr{L}$ defines a filtration of $P^n(\mathscr{L})$ for which

$$\begin{aligned} \operatorname{Gr} P^n(\mathscr{L}) &\simeq \operatorname{Gr} P^n(\mathscr{O}) \otimes \mathscr{L} \\ \operatorname{Gr}^i P^n(\mathscr{L}) &\simeq \Omega^{\otimes i} \otimes \mathscr{L} \qquad (\text{for } 0 \leqslant i \leqslant n). \end{aligned} \tag{4.2.3}$$

Recall that we define, by induction on n, a *differential operator of order* $\leqslant n$ as being a morphism $A : \mathscr{M} \to \mathscr{N}$ of abelian sheaves such that

$$\begin{cases} A \text{ is } \mathscr{O}\text{-linear} & \text{for } n = 0\,; \\ [A, f] \text{ is of order } \leqslant m \text{ for every local section } f \text{ of } \mathscr{O} & \text{for } n = m+1. \end{cases}$$

For every local section s of $\mathscr{L}$, $p_2^* s$ defines a local section $\mathrm{D}^n(s)$ of $P^n(\mathscr{L})$ (4.2.2). The $\mathbb{C}$-linear sheaf morphism $\mathrm{D}^n : \mathscr{L} \to P^n(\mathscr{L})$ is the universal differential operator of order $\leqslant n$ with domain $\mathscr{L}$.

Definition 4.3.

(i) A *linear homogeneous* n^{th}*-order differential equation* on $\mathscr{L}$ is an $\mathscr{O}_X$-homomorphism $E : P^n(\mathscr{L}) \to \Omega^{\otimes n} \otimes \mathscr{L}$ that induces the identity on the submodule $\Omega^{\otimes n} \otimes \mathscr{L}$ of $P^n(\mathscr{L})$.

(ii) A local section s of $\mathscr{L}$ is a *solution* of the differential equation E if $E(\mathrm{D}^n(s)) = 0$.

In fact, I have cheated with this definition, in that I have only considered equations that can be put in the "resolved" form (4.1.1). *p. 25*

4.4. Suppose that $\mathscr{L} = \mathscr{O}$, and let x be a local coordinate on X. The choice of x allows us to identify $p^k(\mathscr{O})$ with $\mathscr{O}^{[0,k]}$, with the arrow D^k becoming

$$\begin{aligned} \mathrm{D}^k : \mathscr{O} &\to P^k(\mathscr{O}) \simeq \mathscr{O}^{[0,k]} \\ f &\mapsto (\partial_x^i f)_{0 \leqslant i \leqslant k}. \end{aligned}$$

The choice of x also allows us to identify Ω^1 with $\mathscr{O}$, so that the n^{th}-order differential equation can be identified with a morphism $E \in \operatorname{Hom}(\mathscr{O}^{[0,n]}, \mathscr{O})$, and, as such, has coordinates $(b_i)_{0 \leqslant i \leqslant n}$ with $b_n = 1$. The solutions of E are then exactly the (holomorphic) functions f that satisfy

$$\sum_{i=0}^{n} b_i(x) \partial_x^i f = 0 \qquad (\text{with } b_n = 1). \tag{4.4.1}$$

The existence and uniqueness theorem for solutions of the Cauchy problem in (4.4.1) implies the following.

Theorem 4.5. (Cauchy). *Let X and $\mathscr{L}$ be as in (4.2), and let E be an n^{th}-order differential equation on $\mathscr{L}$. Then*

(i) the subsheaf of $\mathscr{L}$ given by solutions of E is a local system $\mathscr{L}^E$ of rank n on X ; and

(ii) the canonical arrow $\mathrm{D}^{n-1}\colon \mathscr{L}^E \to P^{n-1}(\mathscr{L})$ induces an isomorphism

$$\mathscr{O} \otimes_{\mathbb{C}} \mathscr{L}^E \xrightarrow{\sim} P^{n-1}(\mathscr{L}).$$

In particular, it follows from (4.5.ii) and from (2.17) that E defines a canonical connection on $P^{n-1}(\mathscr{L})$, whose horizontal sections are the images under D of solutions of E.

4.6. To a differential equation E on $\mathscr{L}$, we have thus associated

a) a holomorphic vector bundle $\mathscr{V}$ endowed with a connection (which is automatically integrable) **!TO-DO!** ; and

b) a surjective homomorphism $\lambda\colon \mathscr{V} \to \mathscr{L}$ (by $i=0$ in (4.2.3)).

Furthermore, the solutions of E are exactly the images under λ of the horizontal sections of $\mathscr{V}$. This is just another way of expressing how to obtain (4.1.2) from (4.1.1). *p. 26*

4.7. Let $\mathscr{V}$ be a rank-n vector bundle on X, endowed with a connection with covariant derivative ∇. Let v be a local section of $\mathscr{V}$, and w a vector field on X that doesn't vanish at any point. We say that v is *cyclic* if the local sections $(\nabla_w)^i(v)$ of $\mathscr{V}$ (for $0 \leqslant i < n$) form a basis of $\mathscr{V}$. This condition does not depend on the choice of w, and if f is an invertible holomorphic function, then v is cyclic if and only if fv is cyclic. In fact, we can show, by induction on i, that $(\nabla_{gw})^i(fv)$ lies in the submodule of $\mathscr{V}$ generated by the $(\nabla_w)^j(v)$ (for $0 \leqslant j \leqslant i$).

If $\mathscr{L}$ is an invertible module, then we say that a section v of $\mathscr{V} \otimes \mathscr{L}$ is cyclic if, for every every local isomorphism between $\mathscr{L}$ and $\mathscr{O}$, the corresponding section of $\mathscr{V}$ is cyclic. This applies, in particular, to sections v of $\underline{\mathrm{Hom}}(\mathscr{V}, \mathscr{L}) = \mathscr{V}^\vee \otimes \mathscr{L}$.

Lemma 4.8. *With the hypotheses and notation of (4.6), λ is a cyclic section of $\underline{\mathrm{Hom}}(\mathscr{V}, \mathscr{L})$.*

Proof. The problem is local on X; we can reduce to the case where $\mathscr{L} = \mathscr{O}$ and where there exists a local coordinate x.

We use the notation of (4.4). A section (f^i) of $P^{n-1}(\mathscr{O}) \simeq \mathscr{O}^{[0,n-1]}$ is horizontal if and only if it satisfies

$$\begin{cases} \partial_x f^i = f^{i+1} & \text{(for } 0 \leqslant i \leqslant n-2) \\ \partial_x f^{n-1} = -\sum_{i=0}^{n-1} b_i f^i. \end{cases}$$

This gives us the coefficients of the connection: the matrix of the connection is

$$\begin{pmatrix} 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 & & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & 0 & -1 \\ b^0 & b^1 & \cdots & \cdots & b^{n-2} & b^{n-1} \end{pmatrix} \tag{4.8.1}$$

p. 27

In the chosen system of coordinates, $\lambda = e^0$, and we calculate that

$$\nabla_x^i \lambda = e^i \qquad (\text{for } 0 \leqslant i \leqslant n-1)$$

which proves (4.8). □

Proposition 4.9. *The construction in (4.6) establishes an equivalence between the following categories, where we take morphisms to be isomorphisms:*

a) the category of invertible sheaves on X endowed with an n^{th}-order differential equation (4.3) ; and

b) the category of triples consisting of a rank-n vector bundle $\mathscr{V}$ endowed with a connection, an invertible sheaf $\mathscr{L}$, and a cyclic homomorphism $\lambda\colon \mathscr{V} \to \mathscr{L}$.

Proof. We will construct a functor that is quasi-inverse to that in (4.6). Let $\mathscr{V}$ be a vector bundle with connection, and λ a homomorphism from $\mathscr{V}$ to an invertible sheaf $\mathscr{L}$. We denote by V the local system of horizontal sections of $\mathscr{V}$. For every $\mathscr{O}$-module $\mathscr{M}$, we have (2.17)

$$\operatorname{Hom}_{\mathscr{O}}(\mathscr{V}, \mathscr{M}) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{C}}(V, \mathscr{M}).$$

In particular, we define a map γ^k from $\mathscr{V}$ to $P^k(\mathscr{L})$ by setting, for any *horizontal section v* of $\mathscr{V}$,

$$\gamma^k(v) = \mathrm{D}^k(\lambda(v)). \tag{4.9.1}$$

Lemma 4.9.2. *The homomorphism λ is cyclic if and only if*

$$\gamma^{n-1}\colon \mathscr{V} \to P^{n-1}(\mathscr{L})$$

is an isomorphism.

The problem is local on X. We can restrict to the case where $\mathscr{L} = \mathscr{O}$ and where we have a local coordinate x. With the notation of (4.4), the morphism γ^k then admits the morphisms $\partial_x^i \lambda = \nabla_x^i$ (for $0 \leqslant i \leqslant k$) as coordinates. For $k = n-1$, these form a basis of $\operatorname{Hom}(\mathscr{V}, \mathscr{O})$ if and only if γ^{n-1} is an isomorphism.

For $k \geqslant l$, the diagram p. 28

$$\begin{array}{ccc} & \mathscr{V} & \\ {}^{\gamma^k}\swarrow & & \searrow^{v^l} \\ P^k(\mathscr{L}) & \xrightarrow{\pi_{l,k}} & P^k(\mathscr{L}) \end{array} \tag{4.9.3}$$

commutes; if λ is cyclic, then this, along with (4.9.2), implies that $\gamma^n(v)$ is locally a direct factor, of codimension 1 in $P^n(\mathscr{L})$, and admits $\omega^{\otimes n} \otimes \mathscr{L} \simeq \mathrm{Ker}(\pi_{n-1,n})$ as a complement. There thus exists exactly one n^{th}-order differential equation

$$E : P^n(\mathscr{L}) \to \Omega^{\otimes n} \otimes \mathscr{L}$$

on $\mathscr{L}$ such that $E \circ \gamma^n = 0$.

By (4.9.1), if v is a horizontal section of $\mathscr{V}$, then $E\mathrm{D}^n\lambda v = E\gamma^n v = 0$, and so λv is a solution of E. We endow $P^{n-1}(\mathscr{L})$ with the connection (4.6) defined by E. If v is a horizontal section of $\mathscr{V}$, then $\gamma^{n-1}(v) = \mathrm{D}^{n-1}\lambda v$, where λv is a solution of E, and $\gamma^{n-1}(v)$ is thus horizontal. We thus deduce that γ^{n-1} is compatible with the connections. A particular case of (4.9.3) shows that the diagram

$$\begin{array}{ccc} \mathscr{V} & \xrightarrow{\gamma^{n-1}} & P^{n-1}(\mathscr{L}) \\ {}_{\lambda}\searrow & & \swarrow_{\textbf{!TO-DO!}} \\ & \mathscr{L} & \end{array}$$

commutes, whence we have an isomorphism between $(\mathscr{V}, \mathscr{L}, \lambda)$ and the triple given by (4.6) applied to $(\mathscr{L}, E)$. The functor

$$(\mathscr{V}, \mathscr{L}, \lambda) \mapsto (\mathscr{L}, E)$$

is thus quasi-inverse to the functor in (4.6). □

4.10. We now summarise the relations between two systems $(\mathscr{V}, \mathscr{L}, \lambda)$ and $(\mathscr{L}, E)$ that correspond under (4.6) and (4.9).

We have homomorphisms $\gamma^k : \mathscr{V} \to P^k(\mathscr{L})$, such that

(4.10.1) $\gamma^k(v) = \mathrm{D}^k\lambda v$ for v horizontal ; p. 29

(4.10.2) $\gamma^0 = \lambda$ and $\pi_{l,k}\gamma^k = \gamma^l$;

(4.10.3) γ^{n-1} is an isomorphism (λ is cyclic) ;

(4.10.4) $E\gamma^n = 0$; and

(4.10.5) λ induces an isomorphic between the local system V of sections of $\mathscr{V}$ and the local system $\mathscr{L}^E$ of solutions of E.

I.5 Second-order differential equations

In this section, we specialise the results of (4) to the case where $n = 2$, and we express certain results given in R.C. Gunning [11] in a more geometric form.

5.1. Let S be an analytic space, and let $q\colon X_2 \to S$ be an analytic space over S that is locally isomorphic to the finite analytic space over S defined by the $\mathcal{O}_S$-algebra $\mathcal{O}_S[T]/(T^3)$.

The fact that the group PGL_2 acts **!TO-DO!** on $\mathbb{P}^1$ has the following infinitesimal analogue.

Lemma 5.2. *Under the hypotheses of (5.1), let u and v be S-immersions of X_2 into $\mathbb{P}^1_S$, i.e.*

$$X_2 \underset{v}{\overset{u}{\rightrightarrows}} \mathbb{P}^1_S.$$

Then there exists exactly one ***!TO-DO!*** *(S-automorphism) of $\mathbb{P}^1_S$ that sends u to v.*

Proof. The problem is local on S, which allows us to suppose that X_2 is defined by the $\mathcal{O}_S$-algebra $\mathcal{O}_S[T]/(T^3)$, and that $u(X_2)$ and $v(X_2)$ are contained inside the same affine line, say, $\mathbb{A}^1_S$. By translation, we can assume that $u(0) = v(0) = 0$. We must then prove the existence and uniqueness of a **!TO-DO!** $p(x) = (ax+b)(cx+d)$ that satisfies $p(0) = 0$, with first derivative $p'(0) \neq 0$ and given second derivative $p''(0)$. We have that $b = 0$, and p can be written uniquely in the form p. 30

$$\begin{aligned} p(x) &= e\frac{x}{1-fx} \qquad (\text{for } e \neq 0) \\ &= ex + efx^2 \quad \mod x^3. \end{aligned}$$

The claim then follows immediately. □

5.3. By (5.2), there exists exactly (up to isomorphism) one pair (u,P) consisting of a projective line P on S (with structure group $\mathrm{PGL}_2(\mathcal{O}_S)$) and an S-immersion u of X_2 into P. We call P the *osculating projective line of* X_2.

Let X be a smooth curve. Let X_2 the second infinitesimal neighbourhood of the diagonal of $X \times X$, and let q_1 and q_2 be the two projections from X_2 to X.

The morphism $q_1\colon X_2 \to X$ is of the type considered in (5.1).

Definition 5.4. We define the *osculating projective line bundle of* X, denoted by P_{tg}, to be the osculating projective line bundle of $q_1\colon X_2 \to X$.

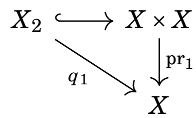
By definition, we thus have a canonical commutative diagram

$$\begin{array}{ccc} X_2 & \hookrightarrow & P_{\mathrm{tg}} \\ & {\scriptstyle q_1}\searrow & \downarrow \\ & & X \end{array} \tag{5.4.1}$$

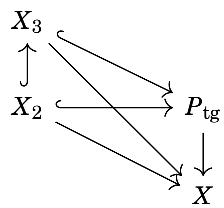
and, in particular, P_{tg} is endowed with a canonical section e, which is the image of the diagonal section of X_2, and we have that

$$e^*\Omega^1_{P_{\mathrm{tg}}/X} \simeq \Omega^1_X. \tag{5.4.2}$$

5.5. If X is a projective line, then $\mathrm{pr}_1 : X \times X \to X$ is a projective bundle on X such that P_{tg} can be identified with the constant projective bundle of fibre X on X endowed with the inclusion homomorphism of X_2 into $X \times X$, i.e.



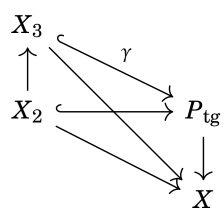
In this particular case, we have a canonical commutative diagram p. 31



Now let X be an arbitrary smooth curve.

Definition 5.6. *(Local version).* A *projective connection* on X is a sheaf on X of germs of local isomorphisms from X to $\mathbb{P}^1$, which is a principal homogeneous sheaf (i.e. a torsor) for the constant sheaf of groups with value $\mathrm{PLG}_2(\mathbb{C})$.

If X is endowed with a projective connection, then every local construction on $\mathbb{P}^1$ that is invariant under the projective group can be transported to X; in particular, the construction in (5.5) gives us a morphism γ that fits into a commutative diagram:



(5.6.1)

It is not difficult to show that such a morphism γ is defined by a unique projective connection (a proof of this will be given in (5.10)), and so Definition (5.6) is equivalent to the following.

Definition 5.6 bis. *(Infinitesimal version).* A *projective connection* on X is a morphism $\gamma : X_3 \hookrightarrow P_{\mathrm{tg}}$ that makes the diagram in (5.6.1) commute.

Intuitively, giving a projective connection (the infinitesimal version) allows us to define **!TO-DO!**

5.7. Set $\Omega^{\otimes n} = (\Omega^1_X)^{\otimes n}$ (4.2). The sheaf of ideals on X_3 that defines X_2 is canonically isomorphic to $\Omega^{\otimes 3}$, and is annihilated by the sheaf of ideals that defines the diagonal. Also, if Δ is the diagonal map, then, by (5.4.2), we have that p. 32

$$\Delta^* \gamma^* \Omega^1_{P_{\mathrm{tg}}/X} \simeq \Omega^1.$$

We thus deduce that the set of X-homomorphisms from X_3 to P_{tg} that induce the canonical homomorphism from X_2 to P_{tg} is either empty, or a principal homogeneous space for

$$\mathrm{Hom}_X(\Delta^*\gamma^*\Omega^1_{P_{\mathrm{tg}}/X},\Omega^{\otimes 3}) = \mathrm{Hom}_X(\Omega^1,\Omega^{\otimes 3}) = \mathrm{H}^0(X,\Omega^{\otimes 2}).$$

If we replace X by a small enough open subset, then this set is non-empty:

Proposition 5.8. *Projective connections of open subsets of X form a principal homogeneous sheaf (i.e. a torsor) for the sheaf $\Omega^{\otimes 2}$.*

Proof. If η is a section of $\Omega^{\otimes 2}$, and $\gamma_1\colon X_3 \to P_{\mathrm{tg}}$ is a projective connection, then the connection $\gamma_2 = \gamma_1 + \eta$ is defined, for any function f on P_{tg}, by

$$\gamma_2^* f = \gamma_1^* f + \eta \cdot e^* \mathrm{d}f \tag{5.8.1}$$

(modulo the identification of $\Omega^{\otimes 3}$ with an ideal of $\mathscr{O}_{X_3}$). □

5.9. Let $f\colon X \to Y$ be a homomorphism between smooth curves endowed with projective connections γ_X and γ_Y, and suppose that f is a local isomorphism (i.e. $\mathrm{d}f \neq 0$ at all points). Set

$$\theta f = f^*\gamma_Y - \gamma_X \in \Gamma(X,\Omega^{\otimes 2}).$$

For a composite map $g \circ f\colon X \xrightarrow{f} Y \xrightarrow{g} Z$, we trivially have that

$$\begin{gathered}\theta(g\circ f) = \theta(f) + f^*\theta(g),\\ \text{whence } \theta(f^{-1}) = -f^*\theta(f).\end{gathered} \tag{5.9.1}$$

Suppose that X and Y are open subsets of $\mathbb{C}$, and endowed with the projective connection induced by that of $\mathbb{P}^1(\mathbb{C})$. Denoting by x the injection from X into $\mathbb{C}$, we then have that *p. 33*

$$\theta f = \frac{f'(f'''/6) - (f''/2)^2}{(f')^2}\mathrm{d}x^{\otimes 2}. \tag{5.9.2}$$

To prove this, we identify, using (5.5), the projective double tangent bundle to X or Y with the constant projective bundle. The morphism $\delta f\colon P_{\mathrm{tg},X} \to P_{\mathrm{tg},Y}$ induced by f can be written as

$$\delta f\colon (x,p) \mapsto \left(f(x), f(x) + \frac{f'(x)(p-x)}{1-\frac{1}{2}(f''(x)/f'(x))(p-x)} \right).$$

Consider the diagram

$$\begin{array}{ccc} X_3 & \xrightarrow{f} & Y_3 \\ \downarrow & & \downarrow \\ P_{\mathrm{tg},X} & \xrightarrow{\delta f} & P_{\mathrm{tg},Y} \end{array}$$

Then $\theta(f)$ describes the lack of commutativity of the diagram, i.e. the difference between the jets

$$(x, x+\varepsilon) \mapsto \left(f(x), f(x) + f'(x)\varepsilon + f''(x)\frac{\varepsilon^2}{2} + f'''(x)\frac{\varepsilon^3}{6} \right) \qquad (\text{mod. } \varepsilon^4 = 0)$$

and

$$(x, x+\varepsilon) \mapsto \left(f(x), f(x) + \frac{f'(x)\varepsilon}{1 - \frac{1}{2}(f''(x)/f'(x))\varepsilon} \right)$$
$$= \left(f(x), f(x) + f'(x)\varepsilon + f''(x)\frac{\varepsilon^2}{2} + \frac{1}{4}(f''(x)^2/f'(x))\varepsilon^3 \right).$$

We thus have that

$$\theta(f) = \left(\frac{1}{6} f'''(x) - \frac{1}{4} f''(x)^2/f'(x) \right) \mathrm{d}x^{\otimes 3} \mathrm{d}f^{\otimes -1},$$

and (5.9.2) then follows.

Equation (5.9.2) shows that $6\theta f$ is the classical *Schwarz derivative* of f.

If a map f from $X \subset \mathbb{C}$ to $\mathbb{P}^1(\mathbb{C})$ is described by projective coordinates $f = (g, h)$, then

$$\theta(f) = \frac{\begin{vmatrix} g & g' \\ h & h' \end{vmatrix} \left(\begin{vmatrix} g & g'''/6 \\ h & h'''/6 \end{vmatrix} + \begin{vmatrix} g' & g'/2 \\ h' & h'/2 \end{vmatrix} \right) - \begin{vmatrix} g & g''/2 \\ h & h''/2 \end{vmatrix}^2}{\begin{vmatrix} g & g' \\ h & h' \end{vmatrix}^2} \tag{5.9.3}$$

To prove (5.9.3), the simplest method is to note the following. p. 34

(i) The right-hand side of (5.9.3) is invariant under a linear substitution of constant coefficients L acting on g and h: the numerator and denominator $\det(L)^2$.

(ii) The right-hand side of (5.9.3) is invariant under the substitution

$$(g, h) \mapsto (\lambda g, \lambda h).$$

Denoting the determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ by $\det(a, b)$, we have that

$$\begin{aligned}
\det(g, (\lambda g)') &= \det(\lambda g, \lambda g' + \lambda' g) \\
&= \lambda^2 \det(g, g') \\
\det(\lambda g, (\lambda g)''/2) &= \lambda^2 \det(g, g''/2) + \lambda\lambda' \det(g, g') \\
\det(\lambda g, (\lambda g)'''/6) &= \lambda^2 \det(g, g'''/6) + \lambda\lambda' \det(g, g''/2) \\
&\quad + (\lambda\lambda''/2) \det(g, g') \\
\det((\lambda g)', (\lambda g)''/2) &= \lambda^2 \det(g', g''/2) - (\lambda\lambda''/2) \det(g, g') \\
&\quad + (\lambda')^2 \det(g, g') + \lambda\lambda' \det(g, g''/2).
\end{aligned}$$

The new denominator D_λ (resp. numerator N_λ) is thus given in terms of the old denominator D (resp. numerator N) by

$$D_\lambda = \lambda^4 D$$

$$N_\lambda = \lambda^4 N + \lambda^4 \det(g,g')\cdot \begin{bmatrix} \Big(\lambda'/\lambda \det(g,g''/2) + (\lambda''/2\lambda)\det(g,g')\Big) \\ +\Big(-(\lambda''/2\lambda)\det(g,g') + (\lambda'/\lambda)^2\det(g,g') + (\lambda'/\lambda)\det(g,g'')\Big) \\ -\Big(2(\lambda'/\lambda)\det(g,g''/2) + (\lambda'/\lambda)^2\det(g,g')\Big) \end{bmatrix}$$

and $N_\lambda/D_\lambda = N/D$.

With these variance properties agreeing with those of the left-hand side of (5.9.3), it suffices to prove (5.9.3) in the particular case where $h = 1$. The equation then reduces to (5.9.2).

We will only need to use the fact that $\theta(f)$ can be expressed in terms of **!TO-DO!** : we have, for $Z_i \sim Z$,

$$\frac{(f(Z_1),f(Z_2),f(Z_3),f(Z_4))}{(Z_1,Z_2,Z_3,Z_4)} - 1 = \theta(f)(Z_1-Z_2)(Z_3-Z_4) + \mathscr{O}((Z_i-Z)^3). \tag{5.9.4}$$

p. 35

5.10. The differential equation $\theta(f) = 0$ (for $f : X \to \mathbb{P}^1(\mathbb{C})$ with non-zero first derivative) is a third-order differential equation. It thus admits ∞^3 solutions, locally, and these solutions are permuted amongst themselves by the projective group (since the group transitively permutes the Cauchy data: (5.2)). The set of solutions is thus a projective connection (the local version, (5.6)). This construction is inverse to that which associates, to any projective connection in the sense of (5.6), a projective connection in the sense of (5.6bis).

5.11. Let X be a smooth curve, $\mathscr{L}$ an invertible sheaf on X, and E a second-order ordinary differential equation on $\mathscr{L}$. We have seen, in (4.5), that E defines a connection on the bundle $P^1(\mathscr{L})$ of first-order jets of sections of $\mathscr{L}$, and we obtain from it a connection on

$$\bigwedge^2 P^1(\mathscr{L}) \simeq \mathscr{L}\otimes\Omega^1(\mathscr{L}) = \Omega^1\otimes\mathscr{L}^{\otimes 2}.$$

If X is a compact connected curve of genus g, then the bundle $\Omega^1\otimes\mathscr{L}^{\otimes 2}$ is thus necessarily of degree 0, and we have that

$$\deg(\mathscr{L}) = 1 - g.$$

Let V be the rank-2 local system of solutions of E; we have (4.5) that $\mathscr{O}\otimes V \xrightarrow{\sim} P^1(\mathscr{L})$, and the linear form $\lambda : P^1(\mathscr{L}) \to \mathscr{L}$ defines a section λ_0 of the projective bundle associated to the vector bundle $P^1(\mathscr{L})$.

Locally on X, V is isomorphic to the constant local system $\underline{\mathbb{C}^2}$; the choice of an isomorphism $\sigma : V \to \underline{\mathbb{C}^2}$ identifies λ_0 with a map $\lambda_{0,\sigma}$ from X to $\mathbb{P}^1(\mathbb{C})$; by (4.8), the differential of this map is everywhere non-zero, and so $\lambda_{0,\sigma}$ allows us to transport the canonical projective connection of $\mathbb{P}^1(\mathbb{C})$ to X. This connection does not depend on the choice of σ, and so the differential equation E defines a projective connection on X.

Proposition 5.12. *Let $\mathscr{L}$ be an invertible sheaf on a smooth curve X. The construction in (5.10) gives a bijection between* *a) the set of second-order ordinary differential equations on $\mathscr{L}$; and* p. 36

b) the set of pairs consisting of a projective connection on X and a connection on $\Omega^1(\mathscr{L}^{\otimes 2})$.

Proof. The problem is local on X; we can thus suppose that X is an open subset of $\mathbb{C}$, and that $\mathscr{L} = \mathscr{O}$. An equation E can then be written as

$$E\colon\ y'' + a(x)y' + b(x)y = 0.$$

If we identify $P^1(\mathscr{L})$ with $\mathscr{O}^2$, then the matrix of the connection (5.10) defined by E on $\wedge^2 P^1(\mathscr{L}) \sim \mathscr{O}$ is then $-a(x)\operatorname{tr} M$, where M is the matrix in (4.8.1).

Let φ be the identity map from X, an open subset of $\mathbb{P}^1(\mathbb{C})$, to itself endowed with the projective connection defined by E. Identifying Ω with $\mathscr{O}$ by means of the given local coordinate, we then have that

$$\theta(\varphi) = \frac{1}{3}b - \frac{1}{12}(a^2 + 2a'). \tag{5.12.1}$$

Indeed, if f and g are two linearly independent solutions of E, then the map with projective coordinates

$$(f,g)\colon X \to \mathbb{P}^1(\mathbb{C})$$

!TO-DO! the projective connection. We have that

$$\begin{aligned} f'' &= -(af' + bf) \\ f''' &= -a(-af' - bf) - bf' - a'f' - b'f \\ &= (a^2 - a' - b)f' + (ab - b')f. \end{aligned}$$

Equation (5.9.3) gives (using the same notation for determinants as before)

$$\begin{aligned} \theta(\varphi) &= \frac{\det(f,f')\left(\frac{1}{6}(a^2 - a' - b)\det(f,f') + \frac{1}{2}b\det(f,f')\right) - (\frac{1}{2}a)^2\det(f,f')^2}{\det(f,f')^2} \\ &= \frac{1}{6}(a^2 - a' - b) + \frac{1}{2}b - \frac{1}{4}a^2 \\ &= \frac{1}{3}b - \frac{1}{12}(a^2 + 2a'). \end{aligned}$$

We conclude by noting that (a,b) is uniquely determined by $(-a, \frac{1}{3}b - \frac{1}{12}(a^2 + 2a'))$, and that, for any holomorphic function g on an open subset U of $\mathbb{C}$, there exists a unique projective connection on U satisfying $\theta(\varphi) = g$, for φ **!TO-DO!** the connection (same proof as for (5.10), or (5.8)). ☐

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I.6 Multiform functions of finite determination

6.1. Let X be a non-empty connected topological space that is both locally path connected and locally simply path connected, and let x_0 be a point of X. We denote by $\pi\colon \widetilde{X}_{x_0} \to X$ the universal cover of (X, x_0), and by $\widetilde{x}_0$ the base point of $\widetilde{X}_{x_0}$.

If $\mathscr{F}$ is a sheaf on X, then we pose:

Definition 6.2. A *multiform section* of $\mathscr{F}$ on X is a global section of the inverse image $\pi^* \mathscr{F}$ of $\mathscr{F}$ on $\widetilde{X}_{x_0}$.

If s is a multiform section of $\mathscr{F}$ on X, then a *determination of s at a point x of X* is an element of the fibre $\mathscr{F}_{(x)}$ of $\mathscr{F}$ at x that is an inverse image of s under a local section of π at x. **!TO-DO! (check)** Each point in $\pi^{-1}(x)$ thus defines a determination of s at x. We define the *base determination* of s at x_0 to be the determination defined by $\widetilde{x}_0$. We define a *determination of s on an open subset U of X* to be a section of $\mathscr{F}$ over U whose **!TO-DO!** at every point of U is a determination of s at that point.

Definition 6.3. We say that $\mathscr{F}$ satisfies the *principle of analytic continuation* if the set where any two local sections of $\mathscr{F}$ agree is always (open and) closed.

Example 6.4. If $\mathscr{F}$ is a coherent analytic sheaf on a complex-analytic space, then $\mathscr{F}$ satisfies the principle of analytic continuation if and only if $\mathscr{F}$ has no **!TO-DO!** components.

Proposition 6.5. *Let X and x_0 be as in (5.1), and let $\mathscr{F}$ be a sheaf of $\mathbb{C}$-vector spaces on X that satisfies the principle of analytic continuation. For every multiform section s of $\mathscr{F}$, the following conditions are equivalent:*

(i) the determinations of s at x_0 generate a finite-dimensional sub-vector space of $\mathscr{F}_{x_0}$; and

(ii) the subsheaf of $\mathscr{F}$ of $\mathbb{C}$-vector spaces generated by the determinations of s is a complex local system (1.1).

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Proof. It is trivial that (ii) implies (i). We now prove that (i) implies (ii). Let x be a point of X at which the determinations of s generate a finite-dimensional sub-vector space of $\mathscr{F}_x$, and let U be a connected open neighbourhood of x over which $\widetilde{X}_{x_0}$ is trivial: $(\pi^{-1}(U), \pi) \simeq (U \times I, \mathrm{pr}_1)$ for some suitable set I. We will prove that, over U, the determinations of s generate a complex local system. Each $i \in I$ defines a determination s_i of s, and, over U, the vector subsheaf of $\mathscr{F}$ generated by the determinations of s is generated by the $(s_i)_{i \in I}$; if this sheaf is constant, then the hypotheses on x implies that it is a local system. We have:

Lemma 6.6. *If a sheaf $\mathscr{F}$ of $\mathbb{C}$-vector spaces on a connected space satisfies the principle of analytic continuation, then the vector subsheaf of $\mathscr{F}$ generated by a family of global sections s_i is a constant sheaf.*

The sections s_i define

$$a : \underline{\mathbb{C}}^{(I)} \to \mathscr{F}$$

with the image being the vector subsheaf $\mathscr{G}$ of $\mathscr{F}$ generated by the s_i. If an equation $\sum_i \lambda_i s_i = 0$ between the s_i holds at a point, then it holds everywhere, by the principle of analytic continuation.

The sheaf $\mathrm{Ker}(a)$ is thus constant subsheaf of $\underline{\mathbb{C}}^{(I)}$, and the claim then follows.

We conclude the proof of (6.5) by noting that, by the above, the largest open subset of X over which the determinations of s generate a local system is closed and contains x_0. □

Definition 6.7. Under the hypotheses of (6.5), a multiform section s of $\mathscr{F}$ is said to be a *finite determination* if it satisfies either of the equivalent conditions of (6.5).

6.8. Under the hypotheses of (6.5), let s be a multiform section of finite determination of $\mathscr{F}$. This section defines *p. 39*

a) the local system V generated by its determinations ;

b) a **!TO-DO!** of V at x_0, say, v_0, corresponding to the base determination of s ; and

c) an inclusion morphism $\lambda\colon V \to \mathscr{F}$.

The triple consisting of V_{x_0}, v_0, and the representation of $\pi_1(X,x_0)$ on V_{x_0} defined by V (1.4) is called the *monodromy* of s. The triple (V, V_0, λ) satisfies the following two conditions.

(6.8.1) v_0 is a cyclic vector of the $\pi_1(X,x_0)$-module V_{x_0}, i.e. it generates the $\pi_1(X,x_0)$-module V_{x_0}.

This simply means that V is generated by the set of determinations of the unique multiform section of V with base determination v_0.

(6.8.2) $\lambda\colon V_{x_0} \to \mathscr{F}_{x_0}$ is injective.

6.9. Let W_0 be a finite-dimensional complex representation of $\pi_1(X,x_0)$, endowed with a cyclic vector w_0. The multiform section s of $\mathscr{F}$ is said to be of *monodromy subordinate to* (w_0, v_0) if it is the finite determination, and if, with the notation of (6.8), there exists a homomorphism of $\pi_1(X,x_0)$-representations of W_0 in V_{x_0} that sends w_0 to v_0. Let W be the local system defined by W_0, and let w be the unique multiform section of w of base determination w_0. It is clear that, under the hypotheses of (6.5), we have

Proposition 6.10. *The function $\lambda \mapsto \lambda(w)$ is a bijection between the set $\mathrm{Hom}_{\mathbb{C}}(W, \mathscr{F})$ and the set of multiform sections of $\mathscr{F}$ with monodromy subordinate to* (W_0, w_0).

Corollary 6.11. *Let X be a reduced connected complex-analytic space endowed with a base point x_0. Let W_0 be a finite-dimensional complex representation of $\pi_1(X,x_0)$ endowed with a cyclic vector w_0, and W the corresponding local system on X, with $\mathscr{W} = \mathscr{O} \otimes_{\mathbb{C}} \mathrm{W}$ being the associated vector bundle, and w the unique multiform section of $\mathscr{W}$ of base determination w_0. Write $\mathscr{W}^\vee$ be the dual vector bundle of $\mathscr{W}$. Then the function* *p. 40*

$$\lambda \mapsto \langle \lambda, w \rangle,$$

from $\Gamma(X, \mathscr{W}^\vee)$ to the set of multiform holomorphic functions on X of monodromy subordinate to (W_0, w_0)*, is a bijection.*

Corollary 6.12. *If X is Stein, then there exist multiform holomorphic functions on X of any given monodromy* (W_0, w_0).

Chapter II

Regular connections

II.1 Regularity in dimension 1

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1.1. Let U be an open neighbourhood of 0 in $\mathbb{C}$, and consider an n^{th}-order differential equation

$$y^{(n)} + \sum_{i=0}^{n-1} a_i(x) y^{(i)} = 0 \tag{1.1.1}$$

where the a_i are holomorphic functions on $U \setminus \{0\}$. We classically say that 0 is a *regular singular point* of (1.1.1) if the functions $x^{n-i}a_i(x)$ are holomorphic at 0. If this is true, then, after multiplying by x^n, we can write (1.1.1) in the form

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^n y + \sum b_i(x)\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^i y = 0 \tag{1.1.2}$$

where the $b_i(x)$ are holomorphic at 0.

In this section, we will translate this idea into the language of connections (cf. (I.4)), and we will establish some of its properties.

The results in this section were taught to me by N. Katz. They are either due to N. Katz (see, most notably, [14, 15]), or classical (see, for example, Ince [13], and Turrittin [25, 26]).

1.2. Let K be a (commutative) field, Ω a rank-1 vector space over K, and $\mathrm{d}\colon K \to \Omega$ a non-trivial derivation, i.e. an non-zero additive map that satisfies the identity

$$\mathrm{d}(xy) = x\mathrm{d}y + y\mathrm{d}x. \tag{1.2.1}$$

Let V be an n-dimensional vector space over K. Then a *connection* on V is an additive map $\nabla\colon V \to \Omega \otimes V$ that satisfies the identity

$$\nabla(xv) = \mathrm{d}x \cdot v + x\nabla v. \tag{1.2.2}$$

If τ is an element of the dual $\Omega^\vee$ of Ω, then we set

$$\partial_\tau(x) = \langle \mathrm{d}x, \tau\rangle \in K, \tag{1.2.3}$$

$$\nabla_\tau(v) = \langle \mathrm{d}v, \tau\rangle \in V. \tag{1.2.4}$$

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We thus have that

(1.2.5) ∂_τ is a derivation;

(1.2.6) $\nabla_\tau(xv) = \partial_\tau(x)\cdot v + x\nabla_\tau v$; and

(1.2.7) $\nabla_{\lambda\tau}(v) = \lambda\nabla_\tau v$.

Let $v \in V$. We can easily show that the vector subspace of V generated by the vectors

$$v, \nabla_{\tau_1} v, \nabla_{\tau_2}\nabla_{\tau_1} v, \ldots, \nabla_{\tau_k}\cdots\nabla_{\tau_1} v$$

(where $\tau_i \neq 0$ in Ω) does not depend on the choice of the $\tau_i \neq 0$, and does not change is we replace v by λv (for some $\lambda \in K^*$). Furthermore, if the last of these vectors is a linear combination of the preceding vectors, then this vector subspace is stable under derivations. We say that v is a *cyclic vector* if, for $\tau \in \Omega$, the vectors

$$\nabla_\tau^i v \qquad (\text{for } 0 \leqslant i \leqslant n)$$

form a basis of V.

Lemma 1.3. *Under the above hypotheses, and if K is of characteristic* 0*, then there exists a cyclic vector.*

Proof. Let $t \in K$ be such that $\mathrm{d}t \neq 0$, and let $\tau = t/\mathrm{d}t \in \Omega^\vee$. Then $\partial_\tau(t^k) = kt^k$.

Let $m \leqslant n$ be the largest integer such that there exists a vector e such that the vectors $\partial_\tau^i e$ (for $0 \leqslant i \leqslant m$) are linearly independent. If $m \neq n$, then there exists a vector f that is linearly independent of the $\partial_\tau^i e$. For any rational number λ and integer k, the vectors

$$\partial_\tau^i(e + \lambda t^k f) \qquad (\text{for } 0 \leqslant i \leqslant m)$$

are linearly dependent, and their exterior product $\omega(\lambda, k)$ is thus zero. We have that

$$\partial_\tau^i(e + \lambda t^k f) = \partial_\tau^i e + \sum_{0\leqslant k\leqslant i} k^j t^k \partial_\tau^{i-j} f.$$

From this equation, we obtain a finite decomposition

$$\omega(\lambda, k) = \sum_{\substack{0\leqslant a\leqslant m\\ 0\leqslant b}} \lambda^a t^{ka} k^b \omega_{a,b}$$

where $\omega_{a,b}$ is independent of λ and k. Since $\omega(\lambda, k) = 0$ for all $\lambda \in \mathbb{Q}$, and since p. 43

$$\omega(\lambda, k) = \sum \lambda^a \omega_a(k)$$

where $\omega_a(k) = t^{ka}(\sum k^b \omega_{a,b}) = t^{ka}\omega'_a(k)$, we have that $\omega_a(k) = \omega'_a(k) = 0$. Since

$$\omega'_a(k) = \sum k^b \omega_{a,b} = 0$$

for all $k \in \mathbb{Z}$, we have that $\omega_{a,b} = 0$. In particular,

$$\omega_{1,m} = e \wedge \partial_\tau^1 e \wedge \ldots \wedge \partial_\tau^{m-1} e \wedge f = 0,$$

and f is then linearly dependent of the $\partial_\tau^i e$ (for $0 \leqslant i \leqslant m$), which contradicts the hypothesis. Thus $m = n$, and so e is a cyclic vector ☐

1.4. Let $\mathcal{O}$ be a discrete valuation ring of *equal characteristic* 0, with maximal ideal $\mathfrak{m}$, residue field $k = \mathcal{O}/\mathfrak{m}$, and field of fractions K. Suppose that $\mathcal{O}$ is endowed with a free rank-1 $\mathcal{O}$-module Ω along with a derivation $\mathrm{d}\colon \mathcal{O} \to \Omega$ that satisfies

1.4.1. *There exists a uniformiser t such that* $\mathrm{d}t$ *generates* Ω.

(For less hyper-generality, see (1.7)).

If t_1 is another uniformiser, then $t_1 = at$ for some $a \in \mathcal{O}^*$, and, by hypothesis, $\mathrm{d}a$ is a multiple of $\mathrm{d}t$, i.e. $\mathrm{d}a = \lambda \mathrm{d}t$. We thus have that

$$\mathrm{d}t_1 = a\mathrm{d}t + \mathrm{d}a \cdot t = (a + \lambda t)\mathrm{d}t$$

and so

1.4.2. *For every uniformiser t,* $\mathrm{d}t$ *generates* Ω.

We denote by

$$v : K^* \to \mathbb{Z}$$

the valuation of K defined by $\mathcal{O}$; we also denote by v the valuation of $\Omega \otimes K$ defined by the lattice Ω. If t is a uniformiser, then

$$v(\omega) = v(\omega/\mathrm{d}t).$$

If $f \in K^*$ with $f = at^n$ (for $a \in \mathcal{O}$), then

$$\mathrm{d}f = \mathrm{d}a \cdot t^n + nat^{n-1}\mathrm{d}t$$

and thus p. 44

(1.4.3) $v(\mathrm{d}f) \leqslant v(f) - 1$; and

(1.4.4) $v(f) \neq 0 \implies v(\mathrm{d}f) = v(f) - 1$.

In particular, d is continuous and extends to $\mathrm{d}\colon \mathcal{O}^\wedge \to \Omega^\wedge$, and the triple $(\mathcal{O}^\wedge, \mathrm{d}, \Omega^\wedge)$ again satisfies (1.4.1).

Lemma 1.5. *If $\mathcal{O}$ is complete, then the triple $(\mathcal{O}, \mathrm{d}, \Omega)$ is isomorphic to the triple $(k[[t]], \partial_t, k[[t]])$.*

Proof. The homomorphisms

$$\mathrm{Gr}(\mathrm{d})\colon \mathfrak{m}^i/\mathfrak{m}^{i+1} \to \mathfrak{m}^{i-1}\Omega/\mathfrak{m}^i\Omega$$

induced by d are linear and bijective (1.4.4). Since $\mathcal{O}$ is complete, $\mathrm{d}\colon \mathfrak{m} \to \Omega$ is surjective, and $\mathrm{Ker}(\mathrm{d}) \xrightarrow{\sim} k$. This gives us a field of representatives that is annihilated by d, and the choice of a uniformiser t gives the desired isomorphism $k[[t]] \xrightarrow{\sim} \mathcal{O}$. □

1.6. If an $\mathcal{O}$-algebra $\mathcal{O}'$ is a discrete valuation ring with a field of fractions K' that is algebraic over K, then the derivation d extends uniquely to $\mathrm{d}\colon K' \to \Omega \otimes_{\mathcal{O}} K'$. Let e be the ramification index of $\mathcal{O}'$ over $\mathcal{O}$, and let t' be a uniformiser of $\mathcal{O}'$. We set

$$\Omega' = 1/(t')^{e-1}\Omega \otimes_{\mathcal{O}} \mathcal{O}'.$$

We can easily show, using (1.6), that the triple $(\mathcal{O}', \mathrm{d}, \Omega')$ again satisfies (1.4.1).

1.7. We will mostly be interested in the following examples. Let X be a non-singular complex algebraic curve, and let $x \in X$. We choose one of the following:

(1.7.1) $\mathscr{O} = \mathscr{O}_{x,X}$ (the local ring for the Zariski topology), $\Omega = (\Omega^1_{X/\mathbb{C}})_x$, and d = the differential ;

(1.7.2) $\mathscr{O} = \mathscr{O}_{x,X^{\mathrm{an}}}$ (the local ring of germs at x of holomorphic functions), $\Omega = (\Omega^1_{X^{\mathrm{an}}/\mathbb{C}})_{(x)}$, and d = the differential ; or

(1.7.3) the common completion of (1.7.1) and (1.7.2).

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1.8. Under the hypotheses of (1.4), let V be a finite-dimensional vector space over K, and V_0 a lattice in V, i.e. a free sub-$\mathscr{O}$-module of V such that $KV_0 = V$. For every homomorphism $e\colon \mathscr{O}^n \to V$, we define the *valuation $v(e)$ of e* to be the largest integer m such that $e(\mathscr{O}^n) \subset \mathfrak{m}^m V_0$. If V_0 and V_1 are two lattices, then there exists an integer s that is independent of e and n and is such that

$$|v_0(e) - v_1(e)| \leqslant s. \tag{1.8.1}$$

Theorem 1.9. [N. Katz]. *Under the hypotheses of (1.4), and with the notation of (1.8), let ∇ be a connection (1.2) on a vector space V of dimension n over K. Then one of the following conditions is satisfied:*

a) For any lattice V_0 in V, any basis $e\colon K^n \xrightarrow{\sim} V$ of V, any differential form with a simple pole ω and $\tau = \omega^{-1} \in \Omega_K$, the numbers $-v(\nabla^i_\tau e)$ are bounded above ; or

b) There exists a rational number $r > 0$, with denominator at most n, such that, for any V_0, e, and τ as above, the family of numbers

$$|-v(\nabla^i_\tau e) - ri|$$

is bounded.

Conditions a) and b) of (1.9) are more manageable in a different form:

Lemma 1.9.1. *Let V_0, τ, and e be as in (1.9). Then, for any given value of r, the bound b) is equivalent to*

$$|\sup_{j\leqslant i}(-v\nabla^i_\tau e) - ri| \leqslant C^{te}. \tag{1.9.2}$$

The bound a) is equivalent to the same bound (1.9.2) for $r = 0$.

Proof. Going from (1.9) to (1.9.2) is clear, as is the converse for $r = 0$. So suppose that (1.9.2) holds true for $r > 0$ and some value C_0 for the constant. We have

$$-v\nabla^i_\tau e - ri \leqslant C_0. \tag{a}$$

We immediately see that there exists a constant k such that

$$-v\nabla^n_\tau(\nabla^i_\tau e) \leqslant -v\nabla^i_\tau e + kn.$$

Thus

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$$\begin{aligned} -C_0 + r(i+n) &\leqslant \sup_{j \leqslant i+n} -v\nabla_\tau^j e \\ &= \sup(\sup_{j\leqslant i} -v\nabla_\tau^j e, -v\nabla_\tau^i e + kn) \\ &\leqslant \sup(C_0 + ri, -v\nabla_\tau^i e + kn) \end{aligned}$$

and, if $-C_0 + r(i+n) > C_0 + ri$, i.e. if $n > 2C_0/r$, then we have

$$-v\nabla_\tau^i e \geqslant (-C_0 - kn - rn) + ri. \tag{b}$$

The inequalities (a) and (b) then imply the inequality of the form (1.9):

$$|-v\nabla_\tau^i e - ri| \leqslant C_0 + kn + rn.$$

□

Lemma 1.9.3. *Let* (V_0, τ_0, e_0) *and* (V_1, τ_1, e_1) *be two systems as in (1.9). Then*

$$|\sup_{j\leqslant i}(-v_1\nabla_{\tau_1}^j e_1) - \sup_{j\leqslant i}(-v_0\nabla_{\tau_0}^j e_0)| \leqslant C^{te}.$$

Proof. It suffices to establish an inequality like (1.9.3) when we change only one of the data V_0, τ_0, or e. The case where we change only the reference lattice V_0 follows from (1.8.1).

We will systematically use the fact that, for $f \in K$, we have, by (1.4.3),

$$v(\partial_{\tau_i} f) \geqslant v(f) \tag{1.9.4}$$

(using the notation of (1.2.3)).

If e and f are two bases, then $e = fa$ for some $a \in \mathrm{GL}_n(K)$, whence

$$\nabla_\tau^i(e) = \sum_j \binom{i}{j} \nabla_\tau^j(f) \cdot \nabla_\tau^{i-j} a,$$

and, by (1.9.4),

$$v(\nabla_\tau^i(e)) \geqslant \inf_{j\leqslant i} v(\nabla_\tau^j f) + C^{te}$$

whence

$$\sup_{j\leqslant i}(-v\nabla_\tau^j e) - \sup_{j\leqslant i}(-v\nabla_\tau^j f) \leqslant C^{te}.$$

Reversing the roles of e and f, we similarly have that

$$\sup_{j\leqslant i}(-v\nabla_\tau^j f) - \sup_{j\leqslant i}(-v\nabla_\tau^j e) \leqslant C^{te}$$

whence the inequality (1.9.3) for a change of basis. *p. 47*

If τ and ω are vectors as in (1.9), then $\sigma = f\tau$, with f invertible, whence

$$\nabla_\sigma = f\nabla_\tau \qquad (\text{with } f \in \mathscr{O}^*)$$

and we can show by induction that

$$\nabla_\sigma^i = \sum_{j \leqslant i} \varphi_j \nabla_\tau^j \qquad (\text{with } \varphi_j \in \mathcal{O}).$$

From this we deduce that

$$v\nabla_\sigma^i(e) \geqslant \inf_{j \leqslant i} v\nabla_\tau^j(e),$$

whence

$$\sup_{j \leqslant i}(-v\nabla_\sigma^j(e)) \leqslant \sup_{j \leqslant i}(-v\nabla_\tau^j(e)).$$

Reversing the roles of σ and τ, we thus conclude that

$$\sup_{j \leqslant i}(-v\nabla_\sigma(e)) = \sup_{j \leqslant i}(-v\nabla_\tau^j(e)). \tag{1.9.5}$$

□

By (1.9.1) and (1.9.3) it suffices, to prove (1.9), to prove an upper bound of the form (1.9.2) for *one* choice of (V_0, τ, e).

Lemma 1.9.6. *Under the hypotheses of (1.9), let $e: K^n \to V$ be a basis of V, t a uniformiser, ω a differential form presenting a simple pole (i.e. a basis of $t^{-1}\Omega$), set $\tau = \omega^{-1} \in \Omega$, and let $\Gamma = (\Gamma_j^i)$ be the connection matrix in the bases e and ω. Let s and $(r_i)_{1 \leqslant i \leqslant n}$ be rational numbers, and set $r_{i,j} = s + r_i - r_j$, and suppose that*

$$-v(\Gamma_j^i) \leqslant r_{i,j}.$$

Finally, let $\gamma \in \mathrm{M}_n(k)$ be the matrix whose coefficients are "$t^{r_{i,j}}\Gamma_j^i \mod m$", i.e.

$$\gamma_j^i = \begin{cases} 0 & \text{if } -v(\Gamma_j^i) < r_{i,j}\,; \\ t^{r_{i,j}}\Gamma_j^i \mod m & \text{if } -v(\Gamma_j^i) = r_{i,j}. \end{cases}$$

Suppose that $s \leqslant 0$, or that γ is not nilpotent. Then the upper bound (1.9.2) is satisfied for $r = \sup\{s, 0\}$.

Proof. Let N be an integer such that the $r_i N$ are integers, and set $\mathcal{O}' = \mathcal{O}(\sqrt[N]{t})$. Let K' be the field of fractions of K, $v: K'^* \to \frac{1}{N}\mathbb{Z}$ the valuation of K' that extends v, and Λ the diagonal matrix with coefficients being the t^{-r_i}. p. 48

On $\mathcal{O}'$, let ω' be the basis of $\Omega \otimes K'$ given by the inverse image of ω, let τ' be the corresponding basis of $\Omega'^\vee \otimes K'$, and let $e' = e\Lambda$ be a new basis of $V' = V \otimes K'$. In these bases, the connection matrix is

$$\Gamma' = \Lambda^{-1}\Gamma\Lambda + \Lambda^{-1}\partial_{\tau'}\Lambda.$$

The matrix $\Lambda^{-1}\partial_{\tau'}\Lambda$ has coefficients in $\mathcal{O}'$, and so either

(a) $s \leqslant 0$, and Γ' has coefficients in $\mathcal{O}'$; or

(b) $s > 0$, $-v(\Gamma') = s$, and the "most polar part" γ of Γ' is not nilpotent, so that $-v(\Gamma'^\ell) = \ell s$.

By the definition of Ω' (1.6), ω' presents a simple pole. In case (a), we thus conclude by induction on ℓ that

$$v(\nabla_{\tau'}^{\ell}, e') \geqslant 0.$$

We can prove by induction on m that, in the basis e',

$$\nabla_{\tau'}^{m} = \sum_{0\leqslant k\leqslant n} (\Gamma'^{m-i} + \Delta_i)\partial_\tau^i$$

where Δ_i is the algebraic sum of the products of at most $m-i-1$ factors $\partial_{\tau'}^{\ell}\Gamma$. In particular,

$$\nabla_\tau^m e' = \Gamma'^m + \Delta_m$$

and, in case (b),

$$-v(\nabla_{\tau'}^{m}, e') = ms.$$

This satisfies (1.9.2) on $\mathscr{O}'$ (for suitable bases), and (1.9.6) then follows from (1.9.3). □

Theorem (1.9) follows from the following proposition and from (I.1.3).

Proposition 1.10. *Under the hypotheses of (1.9), let X be a basis of $\Omega^\vee$, t a uniformiser, set $\tau = tX$, and let v be a cyclic vector (1.2) of V. Set* p. 49

$$\nabla_X^n v = \sum_{i<n} s_i \nabla_X^i v$$
$$\nabla_\tau^n v = \sum_{i<n} b_i \nabla_\tau^i v.$$

Then the bound (1.9.2) holds true for

$$\begin{aligned} r &= \sup\left\{0, \sup\{-v(b_i)/n-i\}\right\} \\ &= \sup\left\{0, \sup\{-v(a_i)/n-i\} - 1\right\}. \end{aligned}$$

The same conclusion holds for a cyclic vector v of $V^\vee$.

This proposition gives us a procedure for calculating r for a given vector bundle V with a connection defined by an n^{th} order differential equation (cf. (I.4.8)).

Proof. We have the identities

$$(t\nabla_X)^n - \sum b_i (t\nabla_X)^i = t^n\left(\nabla_X^n - \sum a_i \nabla_X^i\right)$$
$$(t^{-1}\nabla_\tau)^n - \sum a_i (t^{-1}\nabla_\tau)^i = t^{-n}\left(\nabla_\tau^n - \sum b_i \nabla_\tau^i\right).$$

From these identities, we see that

$$a_i = g_{n,i} + \sum_{j\geqslant i} g_{j,i} b_j, \quad v(g_{j,i}) \geqslant i-n$$
$$b_i = h_{n,i} + \sum_{j\geqslant i} h_{j,i} a_j, \quad v(h_{i,j}) \geqslant n-j$$

and, for $i \geqslant 0$,

$$\sup\{0, \sup_{j\geqslant i}\{-v(b_j)\}\} = \sup\{0, \sup_{j\geqslant i}\{-v(a_j) - (n-j)\}\}.$$

The two expressions given for r thus agree.

If $v \in V$ is a cyclic vector, then the matrix of the connection, in the bases $(\nabla_\tau^i v)_{0\leqslant i\leqslant n}$ of V and τ of $\Omega^\vee$, is p. 50

$$\Gamma = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & & b_0 \\ 1 & 0 & 0 & 0 & \cdots & & b_1 \\ 0 & 1 & 0 & 0 & & & \vdots \\ \vdots & & \ddots & \ddots & & & \vdots \\ \vdots & & & 1 & 0 & & b_{n-2} \\ 0 & \cdots & \cdots & \cdots & 1 & & b_{n-1} \end{pmatrix}.$$

If $v \in V^\vee$ is a cyclic vector, then the matrix of the connection, in the basis of V given by the dual of $(\nabla_\tau^i v)_{0\leqslant i\leqslant n}$ and the basis τ of $\Omega^\vee \otimes K$, is

$$\Gamma = -\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & & \vdots \\ 0 & 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & 0 & 1 \\ b^0 & b^1 & \cdots & \cdots & b^{n-2} & b^{n-1} \end{pmatrix}$$

It remains only to apply (1.9.6). For $v \in V$, we take $r_i = -ri$ and $s = r$. For $v \in V^\vee$, we take $r_i = ri$ and $s = r$. In the first (resp. second) case, if $s = r > 0$, then the matrix γ is of the type

$$\gamma = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & * \\ 1 & 0 & 0 & 0 & \cdots & * \\ 0 & 1 & 0 & 0 & & \vdots \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & 1 & 0 & * \\ 0 & \cdots & \cdots & \cdots & 1 & * \end{pmatrix}$$

with one of the coefficients of the last column being non-zero (resp. of the type given by the transpose of this). The coefficients are those of the characteristic polynomial of γ, which is thus not nilpotent for $s > 0$. ☐

Definition 1.11. Under the hypotheses of (1.9), we say that the connection ∇ is *regular* if condition a) of (1.9) is satisfied.

Theorem 1.12. [N. Katz]. *Under the hypotheses of (1.9), we have the following:*

(i) For the connection ∇ to be regular, it is necessary and sufficient for V to admit a basis e such that the matrix of the connection, in this basis, is a matrix of differential forms presenting, at worst, simple poles. p. 51

(ii) For the connection ∇ to be irregular, and to satisfy a bound of the form (1.9.2) for $r = a/b > 0$, it is necessary and sufficient for V to admit a basis e (after a change of rings from $\mathcal{O}$ to $\mathcal{O}' = \mathcal{O}(\sqrt[b]{t})$, and for the natural valuation, with values in $\mathbb{Z}$, of $\mathcal{O}'$) such that the matrix of the connection, in this basis, presents a pole of order $a+1$, and for the polar part of order $a+1$ of this matrix (which is in $\mathrm{M}_n(k)$ and determined up to a scalar multiple) to be non-nilpotent.

Proof. By extension of scalars, the number r such that ∇ satisfies (1.9.2) is multiplied by the ramification index. This leads us to the case where $b = 1$. Conditions (i) and (ii) are then sufficient, by (1.9.6). Conversely, let v be a cyclic vector (1.3), t a uniformiser, and $\tau \in \Omega^\vee$ of valuation 1. Then it follows from the proofs of (1.9.6) and of (1.10) that the basis $e_i = t^{ri}\nabla_\tau^i v$ (for $0 \leqslant i \leqslant \dim V$) satisfies (i) or (ii). ☐

Proposition 1.13.

(i) For every horizontal exact sequence

$$V' \to V \to V'',$$

if the connections on V' and V'' are regular, then the connection on V is regular.

(ii) If the connections on V_1 and V_2 are regular, then the natural connections of

$$V_1 \otimes V_2, \quad \mathrm{Hom}(V_1, V_2), \quad V_1^\vee, \quad \bigwedge^p V_1, \quad \ldots$$

are regular.

(iii) If $\mathcal{O}'$ is a discrete valuation ring with field of fractions k' that is algebraic over the field of fractions K of $\mathcal{O}$, and if $V' = V \otimes_K K'$, then the connection on V' is regular if and only if the connection on V is regular.

Proof. Claim (iii), already utilised in (1.12), follows for example from the calculation in (1.10) and from the fact that the inverse image of a differential form presenting a simple pole again presents a simple pole. p. 52

Claim (ii) follows immediately from criterion (i) in (1.12).

Claim (i) implies that, for every short exact sequence

$$0 \to V' \to V \to V'' \to 0,$$

V is regular if and only if V' and V'' are regular. After an eventual extension of scalars, we choose bases e' and e'' of V' and V'' (respectively) satisfying (i) or (ii) in (1.12). Lift e'' to a family of vectors e''_0 of V. For large enough N, the basis $e' \cup t^{-N} e''_0$ of V will satisfy (i) in (1.12) if e' and e'' satisfy (i) in (1.12), and will satisfy (ii) in (1.12) in the contrary case. □

1.14. Let S be a Riemann surface, $p \in S$, and z a uniformiser at p. We denote by j the inclusion of $S^* = S \setminus \{p\}$ in S. We say that a (holomorphic) vector bundle on S^* is *meromorphic* at p if we have the data of

(i) a vector bundle V on S^* ; and

(ii) an equivalence class of extensions of V to a vector bundle on S, with two extensions V_1 and V_2 being equivalent if there exists an integer n such that $z^n V_1 \subset V_2 \subset z^{-n} V_1 \subset j_* V$.

Such a bundle defines a vector space V_K over the field of fractions K of the local ring $\mathscr{O}_{p,S}$. We talk of a *basis* of V to mean a basis that extends to a basis of one of the permissible extensions of V. It is clear that V admits bases of this type on a neighbourhood of p. A connection ∇ on V is said to be *meromorphic* at p if its coefficients (in any basis of V) are meromorphic at p. Such a connection defines a connection (1.2) on V_K (cf. (1.7.2)). We say that a connection ∇ on V is *regular* at p if it is meromorphic at 0 and if the induced connection on V_K is regular in the sense of (1.11), i.e. if there exists a basis of V close to p in which the matrix of the connection presents at most a simple pole at p (1.12).

1.15. Let D be the open unit disc

$$D = \{z : |z| < 1\}$$

and let $D^* = D \setminus \{0\}$. The group $\pi_1(D^*)$ is cyclic, generated by the loop $t \mapsto \lambda e^{2\pi i t}$ (for *p. 53*
$0 \leqslant t \leqslant 1$). The fundamental groupoid is thus the constant group $\mathbb{Z}$. It acts on any local system on D^*. Using the dictionary (I.2), every vector bundle V with connection is thus endowed with an action of the local fundamental group $\mathbb{Z}$. The generator T of this action is called the *monodromy transformation*.

1.16. Let V be a vector bundle on D, and ∇ a connection on $V|D^*$ that is meromorphic at 0. If e_1 (resp. e_2) is a basis of V in which ∇ is represented by $\Gamma_1 \in \Omega^1(\mathrm{End}(V|D^*))$ (resp. Γ_2), then the difference $\Gamma_1 - \Gamma_2$ is holomorphic at 0. Thus the polar part of Γ does not depend on the choice of e.

Suppose that Γ_i presents only a simple pole at 0, and thus has "polar part" equal to some element γ_i in

$$\mathrm{H}^0\left(\left(\frac{1}{z}\Omega^1/\Omega^1\right) \otimes \underline{\mathrm{End}}(V)\right).$$

The "residue" map $\mathrm{H}^0((1/z)\Omega^1/\Omega^1) \to \mathbb{C}$ then associates to γ_i an endomorphism of the fibre V_0 of V at 0. We call this endomorphism the *residue* of the connection at 0, and denote it by

$$\mathrm{Res}(\Gamma_i) \in \mathrm{End}(V_0).$$

Theorem 1.17. *Under the hypotheses of (1.16), the monodromy transformation T extends to an automorphism of V whose fibre at 0 is given by*

$$T_0 = \exp(-2\pi i \operatorname{Res}(\Gamma)).$$

Proof. We can take $V = \mathcal{O}^n$; the differential equation for the horizontal sections is then

$$\partial_z v = -\Gamma v,$$

and the differential equation for a horizontal basis $e\colon \mathcal{O}^n \to V$ is thus

$$\partial_z e = -\Gamma \circ e. \tag{1}$$

In polar coordinates (r,θ),

$$z = re^{i\theta}$$
$$\mathrm{d}z = rie^{i\theta}\mathrm{d}\theta + \mathrm{d}re^{i\theta},$$

and this equation gives p. 54

$$\partial_\theta e = -ir\Gamma \circ e.$$

Set $\Gamma = \frac{\Gamma_0}{z} + \Gamma_1$, where Γ_0 is constant and Γ_1 is holomorphic. The above equation can then be rewritten as

$$\partial_\theta e = -(ie^{-i\theta}\Gamma_0 + ir\Gamma_1)e.$$

The monodromy transformation at (r,θ) is the value at $(r,\theta+2\pi)$ of the solution of this differential equation which is the identity at (r,θ). As $r \to 0$, the aforementioned solution tends to the solution of the limit equation

$$\partial_\theta e = -ie^{-i\theta}\Gamma_0 \circ e. \tag{2}$$

We thus deduce that T has a limit value as $z \to 0$, for θ fixed, and that this value depends continuously on θ. In particular, T is bounded near 0, and thus extends to an endomorphism T of V on D. We conclude that T has a limit value as $z \to 0$; this value, given by integrating (1), depends only on Γ_0. To calculate this limit value, it suffices to calculate it for an arbitrary connection Γ' that has the same residue as Γ.

For example, we can prove:

Lemma 1.17.1. *Let ∇ be the connection on $\mathcal{O}^n$ given by the matrix $U \cdot \frac{\mathrm{d}z}{z}$ for some $U \in$* $\mathrm{GL}_n(\mathbb{C})$*. Then the general solution of the equation $\nabla e = 0$ is*

$$e = \exp(-\log z \cdot U)f = \textbf{\textit{!TO-DO!}}$$

and thus the monodromy is the automorphism of $\mathcal{O}^n$ given by the constant matrix $\exp(-2\pi i U)$.

Corollary 1.17.2. *Under the above hypotheses, the automorphism* $\exp(-2\pi i \operatorname{Res}(\Gamma))$ *of the fibre of V at 0 is the limit of the conjugates of the monodromy automorphism.*

Note that it is not true in general that T_0 is conjugate to T_x for x close to 0. For example, if ∇ is the connection on $\mathcal{O}^2$ for which

$$\nabla \begin{pmatrix} u \\ v \end{pmatrix} = d\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}\frac{\mathrm{d}z}{z} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix},$$

then the general horizontal section is p. 55

$$\begin{aligned} u &= a \\ v &= az\log z + bz \end{aligned}$$

and the monodromy transform is

$$T = \begin{pmatrix} 1 & 2\pi i z \\ 0 & 1 \end{pmatrix}.$$

However, it follows from 1.17.2 that T *and* T_0 *have the same characteristic polynomial*. See also 5.6. ☐

1.18. Let f be a multiform function on D^*. Let D_1 be the disc D^* minus the "cut" $\mathbf{R}^+ \cap D^*$. We say that f *is of moderate growth at* 0 if all the determinations of f on D_1 grow as $1/r^n$ for some suitable n:

$$f \leqslant A|z|^{-n}.$$

Here we allow n to vary with the determination. It is evident, however, that, for f of moderate growth and of finite determination, there exists some n that works for all the determinations. The fact that f is of moderate growth also implies that the function $f(e^{2\pi i z})$ is of at most exponential order in each vertical strip.

If f is a multiform section of a vector bundle V on D^* that is meromorphic at 0, then we say that f *is of moderate growth at* 0 if its coordinates, in an arbitrary base of V near 0, are of moderate growth.

Theorem 1.19. *Let V be a vector bundle on D^* that is meromorphic at* 0*, endowed with a connection ∇. Then the following conditions are equivalent:*

(i) ∇ *is regular; and*

(ii) the (multiform) horizontal sections of V are of moderate growth at 0.

Proof. (i) $\Longrightarrow$ (ii). Choose, near to 0, an isomorphism $V \sim \mathcal{O}^n$, under which the differential equation for the horizontal sections is of the form p. 56

$$\partial_z v = \Gamma v,$$

where Γ has at most a simple pole at 0. We then have, for $|z| \leqslant \lambda < 1$,

$$|\partial_z v| \leqslant \frac{k}{|z|}|v|$$

and, on D_1 (1.18), this inequality integrates, for $|z| \leqslant \lambda$, to

$$|v| \leqslant \frac{1}{|z|^k} \sup_{|z|=\lambda} |v|.$$

(ii) $\Longrightarrow$ (i). Let T be the monodromy transformation of V, and let $U \in \mathrm{GL}_n(\mathbb{C})$ be a matrix such that $\exp(2\pi i U)$ is conjugate to T. Let V_0 be the vector bundle $\mathscr{O}^n$ endowed with the regular connection with matrix

$$\Gamma = \frac{U}{z}.$$

The bundles V and V_0 have the same monodromy. By the dictionaries in I.1 and I.2, they are thus isomorphic as bundles with connections on D^*. Let

$$\varphi \colon V_0|D^* \to V|D^*$$

be an isomorphism. It suffices to prove that φ is compatible with the structures on V_0 and V of bundles that are meromorphic at 0; this is the case if and only if φ is of moderate growth at 0. Let e be a (multiform) horizontal basis of $V_0|D^*$, and let f be a (multiform) horizontal basis of $V|D^*$.

$$\begin{array}{ccc} V_0 & \longrightarrow & V \\ {\scriptstyle e}\uparrow & & \uparrow{\scriptstyle f} \\ \mathscr{O}^n & \longrightarrow & \mathscr{O}^n \end{array} \tag{i}$$

The morphism f is, by hypothesis, of moderate growth. The morphism e^{-1} has horizontal sections of the regular bundle $V_0^\vee$ as its coordinates, and is thus of moderate growth. The morphism ψ that makes diagram (i) commute is horizontal with respect to the usual connection on $\mathscr{O}^n$, and is thus constant. The composite $\varphi = f\psi e^{-1}$ is thus of moderate growth. □

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Corollary 1.20. *Let V_1 and V_2 be vector bundles on D^* that are meromorphic at 0, and endowed with regular connections ∇_1 and ∇_2 (respectively). Then every horizontal homomorphism $\varphi \colon V_1 \to V_2$ is meromorphic at zero. In particular, V_1 and V_2 are isomorphic if and only if they have the same monodromy.*

Proof. Indeed, φ, thought of as a section of $\underline{\mathrm{Hom}}(V_1, V_2)$, is horizontal, and thus of moderate growth, since the connection on $\underline{\mathrm{Hom}}(V_1, V_2)$ is regular. □

1.21. Let X be a smooth algebraic curve over a field k of characteristic 0, and V a vector bundle on X endowed with a connection

$$\nabla \colon V \to \Omega^1_{X/k}(V).$$

Let $\overline{X}$ be the smooth projective curve given by the completion of X, and $x_\infty \in \overline{X} \setminus X$ a "point at infinity" of X. The local ring $\mathscr{O}_{x_\infty}$, endowed with

$$\mathrm{d} \colon \mathscr{O}_{x_\infty} \to \Omega^1_{x_\infty}$$

satisfies (1.4.1), and V induces a vector space V_K endowed with a connection, in the sense of (1.2), over the field of fractions K of $\mathscr{O}_{x_\infty}$ (which is equal to the field of functions of X in the case when X is connected). We say that the connection on V is *regular at x_∞* if this induced connection on V_K is *regular* in the sense of (1.10).

If $\overline{X}_1$ is an arbitrary curve that contains X as a dense open subset, and if $S \subset \overline{X}_1 \setminus X$, then we say that the connection ∇ is *regular at S* if it is regular at all points of the inverse image of S in $\overline{X}$ (which makes sense, since the normalisation of $\overline{X}_1$ can be identified with an open subset of $\overline{X}$).

Finally, we say that the connection ∇ is *regular* if it is regular at all points at infinity of X.

1.22. If $k = \mathbb{C}$, then every vector bundle V on X can be extended to a vector bundle on the completed curve $\overline{X}$. If V_1 and V_2 are two extensions of V, and if t is a uniformiser at a point $x_\infty \in \overline{X} \setminus X$, then there exists some $N \in \mathbb{N}$ such that, in a neighbourhood of x_∞, the subsheaves V_1 and V_2 of the direct image of V satisfy *p. 58*

$$t^N V_1 \subset V_2 \subset t^{-N} V_1.$$

The bundle V^{an} is thus canonically endowed with a structure that is meromorphic at every $x_\infty \in \overline{X} \setminus X$.

If V is endowed with a connection, then we can immediately see (I.12) that (V, ∇) is regular at $x_\infty \in \overline{X} \setminus X$, in the sense of (1.21), if and only if (V^{an}, ∇) is regular at x_∞, in the sense of (1.14).

[Translator] Theorem 1.23 and Proposition 1.24 have been removed from this edition, due to the following comment from the errata:

> *I thank B. Malgrange for having shown me that the "theorem" in (II.1.23) is false. We incorrectly suppose, in the proof, that the vector field τ has no poles. The statement of (II.1.23) was used in the proof of the key theorem (II.4.1), and only there.*

II.2 Growth conditions

2.1. Let X^* be a separated scheme of finite type over $\mathbb{C}$. By Nagata [20] (see also EGA II, 2nd edition), X^* can be represented by a dense Zariski open subset of a scheme X that is proper over $\mathbb{C}$ (here, proper = complete = compact). Furthermore, if X_1 and X_2 are two "compactifications" of X^*, then there exists a third compactification X_3 along with two commutative diagrams

$$\begin{array}{ccc} X^* & \hookrightarrow & X_3 \\ \| & & \downarrow \\ X^* & \hookrightarrow & X_i \end{array}$$

We can take X_3 to be the scheme-theoretic closure of the diagonal image of X in $X_1 \times X_2$.

This makes schemes over $\mathbb{C}$ much more well behaved at infinity than analytic varieties are. Often, an algebraic object or construction with respect to some scheme X^* can be seen as an analogous analytic object or construction, plus a "growth condition at infinity". *p. 61*

2.2. Let X be a separated complex-analytic space, and Y a closed analytic subset of X. Let $X^* = X \setminus Y$, and let $j \colon X^* \hookrightarrow X$ be the inclusion morphism of X^* into X. In what follows,

X is seen as a "partial compactification" of X^*, with Y being the "at infinity". We will not lose much generality in supposing that X^* is dense in X.

2.3. Suppose that X^* is smooth, and that X admits an embedding into some $\mathbb{C}^n$ (or, more generally, into some smooth analytic space). If j_1 and j_2 are embeddings of X into $\mathbb{C}^{n_i}$, then the two Riemannian structures on X^* induced by $\mathbb{C}^{n_i}$, say $j_1^* g$ and j_2^*, satisfy the following:

For every compact subset K of X, there exist constants $A, B > 0$ such that

$$j_1^* g \leqslant A j_2^* g \leqslant B j_2^* g \leqslant B j_1^* g. \tag{$*$}$$

on $K \cap X^*$.

To see this, we compare $j_i^* g$ with $j_3^* g$, where j_3 is the diagonal embedding of X into $\mathbb{C}^{n_1+n_2}$. Locally on X, we have $j_3 = \alpha \cdot j_i$, where $\alpha \colon \mathbb{C}^{n+i} \to \mathbb{C}^{n_1+n_2}$ is a holomorphic section of pr_i, and the claim then follows.

The compactification X of X^* thus defines an equivalence class under $(*)$ of Riemannian structures on X^*.

Now suppose only that X^* is smooth. A *Riemannian structure g on X^** is said to be *adapted to X* if, for every open subset U of X that admits an embedding into $\mathbb{C}^n$, the restriction $g|U \cap X^*$ is in the class described above, with respect to $U \cap X^* \hookrightarrow U$. This condition is local on X. Using a partition of unity, we can show that there exist Riemannian structures on X^* that are adapted to X; these form an equivalence class under $(*)$.

2.4. We situate ourselves under the hypotheses of (2.2). We have multiple ways of defining the distance from a point of X^* to the infinity Y.

2.4.1. Suppose that Y is defined in X by a finite family of equations $f_i = 0$. We set p. 62

$$d_1(x) = \sum f_i(x)\overline{f_i(x)}.$$

If the functions $d_1'(n)$ and $d_1''(n)$ are obtained by this procedure, then we have:

For every compact subset K of X, there exist constants $A_1, A_2 > 0$ and $\rho_1, \rho_2 > 0$ such that, for all $x \in X^*$,

$$\begin{aligned} d_1'(x) &\leqslant A_1 d_1''(x)^{\rho_1} \\ d_1''(x) &\leqslant A_2 d_1'(x)^{\rho_2}. \end{aligned} \tag{$*$M}$$

Indeed, let $(f_i' = 0)$ and $(f_i'' = 0)$ be two systems of equations for Y. It suffices to verify $(*\mathrm{M})$ locally on X. Locally, by the analytic Nullstellensatz, we know that, for large enough N, the $f_i'^N$ (resp. $f_i''^N$) are linear combinations of the f_i'' (resp. f_i'), and $(*\mathrm{M})$ then follows formally.

2.4.2. Suppose that X admits an embedding $j \colon X \hookrightarrow \mathbb{C}^n$. Let U be a relatively compact open subset of X. In U, we set

$$d_2(x) = d(j(x), j(Y \cap U))$$

where d is the Euclidean distance in $\mathbb{C}^n$.

We can show, as in (2.3), that, if d_2' and d_2'' are obtained by this method, with respect to two different embeddings, then we have:

For every compact subset K of U, there exist constants $A, B > 0$ such that, for all $x \in K \cap X^*$,

$$d_2'(x) \leqslant A d_2''(x) \leqslant B d_2'(x). \tag{$*$R}$$

Furthermore, it follows immediately from the Lojasiewicz inequalities [18, Th. 1, p. 85] that the "distances at infinity" (2.4.1) and (2.4.2) are equivalent in the sense of ($*$M).

Definition 2.5. Under the hypotheses of (2.2), a norm $\|x\|$ on X^* is said to be *adapted to* X if it is a function from X^* to $\mathbf{R}^+$ such that, for every open subset U of X on which Y is defined by a finite family of equations ($f_i = 0$), and for every compact subset K of U, there exist constants $A_1, A_2 > 0$ and $\rho_1, \rho_2 > 0$ such that, for all $x \in K \cap X^*$, we have

$$(1 + \|x\|)^{-1} \leqslant A_1 \left(\sum f_i(x)\overline{f_i(x)}\right)^{\rho_1}$$
$$\sum f_i(x)\overline{f_i(x)} \leqslant A_2 (1 + \|x\|)^{-\rho 2}.$$

These conditions are local on X. We can show, using a partition of unity, that there always exist norms on X^* that are adapted to X. These form an equivalence class under the equivalence relation p. 63

For every compact subset K of X, there exist constants $A_1, A_2 > 0$ and $\rho_1, \rho_2 > 0$ such that, for all $x \in K \cap X^*$,

$$(1 + \|x\|_i) \leqslant A_i (1 + \|x\|_j)^{\rho_i} \tag{$*'$M}$$

for $i = 1, 2$.

Definition 2.6. A function f on X^* is said to be of *moderate growth along Y* if there exists a norm $\|x\|$ on X^* that is adapted to X and such that, for all $x \in X^*$,

$$|f(x)| \leqslant \|x^*\|.$$

This condition is local on X.

2.7. More precise information about the structure at infinity of X^* is necessary in order to reasonably define what a multiform function with moderate growth at infinity on X^* is.

The fundamental example is that of the logarithm function. Denote by $\widetilde{D}^*$ the universal cover of the punctured disc. At an arbitrary point of D^*, the set of determinations of $\log\colon \widetilde{D}^* \to \mathbb{C}$ is not bounded. We only have a bound

$$|\log(z)| \leqslant A(1/|z|)^{\varepsilon}$$

in a subset of $\widetilde{D}^*$ where the argument $\arg(z)$ of z is bounded.

The delicate results of Lojasiewicz used below will only be essential in what follows for trivial cases (cf. (2.20)).

2.8. In [17], Lojasiewicz proves results which are more precise than (2.8.2) given below.

2.8.1. Let X be a separated analytic space. In the following, we understand "semi-analytic triangulation of X" in the following weak sense: a semi-analytic triangulation of X is a set $\mathscr{T}$ of closed semi-analytic subsets of X (the simplices of the triangulation) such that

(a) $\mathscr{T}$ is locally finite and stable under intersection; and

(b) for every $\sigma \in \mathscr{T}$, there exists a homeomorphism γ between σ and a simplex of the form Δ_n such that p. 64

  (b1) the graph $\Gamma \subset \mathbf{R} \times X$ of γ is semi-analytic, and even semi-algebraic in the first variable; and

  (b2) γ sends the set of faces of Δ_n to the set of $\tau \in \mathscr{T}$ contained in σ.

2.8.2. Let X be separated analytic space, and $\mathscr{F}$ a locally finite set of semi-analytic subsets of X. Then, locally on X, there exists a semi-analytic triangulation $\mathscr{T}$ of X such that every $F \in \mathscr{F}$ is the union of the simplices of the triangulation that it contains.

Definition 2.9. Under the hypotheses of (2.2), a subset P of a covering $\pi\colon \widetilde{X}^* \to X^*$ of X^* is said to be *vertical* along Y if there exists a finite family of compact semi-analytic subsets P_i of X such that the $P_i \setminus Y$ are simply connected, and lifts $\widetilde{P_i}$ of the $P_i \setminus Y$ on $\widetilde{X}^*$ such that

$$P \subset \bigcup_i \widetilde{P_i}.$$

2.9.1. If $\mathscr{T}$ is a semi-analytic triangulation of X that induces a semi-analytic triangulation of Y, then a subset P of $\widetilde{X}^*$ is vertical if and only if it is contained in the union of a finite nubmer of lifts of open simplices of $\mathscr{T}$.

2.9.2. If X is a finite union of open subsets U_i, then, for a subset P of $\widetilde{X}^*$ to be vertical, it is necessary and sufficient for P to be a union of subsets $P_i \subset \pi^{-1}(U_i)$ that are vertical along $Y \cap U_i$.

2.9.3. If U is an open subset of X, and P is a vertical subset of $\widetilde{X}^*$, then, for every compact subset K of U, $P \cap \pi^{-1}(K)$ is vertical in $\pi^{-1}(U)$ along $U \cap Y$.

Definition 2.10. Under the hypotheses of (2.2), let $\pi\colon \widetilde{X}^* \to X^*$, and let f be a function on $\widetilde{X}^*$. We say that f is *of moderate growth along* Y if, for every norm $\|x\|$ on X^* that is adapted to X, and for every vertical subset P of $\widetilde{X}^*$, there exist $A, N > 0$ such that, for all $x \in P$, p. 65

$$|f(x)| \leqslant A(1 + \|x\|)^N.$$

This condition is of a local nature on X.

Example 2.11. Let X be the disc, X^* the punctured disc, and $\widetilde{X}^*$ the universal cover of X^*. The multiform functions on X^* (or functions on $\widetilde{X}^*$) defined by $z \mapsto z^\rho$ (for $\rho \in \mathbb{C}$), and also $z \mapsto \log z$, are of moderate growth at the origin.

Lemma 2.12. *Under the hypotheses of (2.2), let $\mathcal{V}$ be a coherent analytic sheaf on X^*, and $\mathcal{V}_1$ and $\mathcal{V}_2$ extensions of $\mathcal{V}$ to a coherent analytic sheaf on X. Then the following conditions are equivalent.*

(i) There exists an extension $\mathcal{V}'$ of $\mathcal{V}$ on X, as well as homomorphisms from $\mathcal{V}'$ to $\mathcal{V}_1$ and to $\mathcal{V}_2$.

(ii) There exists an extension $\mathcal{V}''$ of $\mathcal{V}$ on X, as well as homomorphisms from $\mathcal{V}_1$ and from $\mathcal{V}_2$ to $\mathcal{V}''$.

(iii) The two previous conditions are locally satisfied on Y.

Proof. To prove that (i) $\Longleftrightarrow$ (ii), we take

$$\mathcal{V}'' = (\mathcal{V}_1 \oplus \mathcal{V}_2)/\mathcal{V}'$$

where

$$\mathcal{V}' = \mathcal{V}_1 \cap \mathcal{V}_2.$$

If (i) is locally true, then a global solution is given by the sum of the images of $\mathcal{V}_1$ and $\mathcal{V}_2$ in $j_*\mathcal{V}$, where j is the inclusion of X^* into X. ☐

2.13. We say that two extensions $\mathcal{V}_1$ and $\mathcal{V}_2$ of $\mathcal{V}$ are *meromorphically equivalent* if the conditions of (2.2) are satisfied; we say that a coherent analytic sheaf on X^* is *meromorphic along Y* if it is locally endowed with an equivalence class of extensions of $\mathcal{V}$ to a coherent analytic sheaf on X. If there exists an extension of $\mathcal{V}$ on X that is, locally on Y, meromorphically equivalent to the pre-given extensions, then this extension is unique up to meromorphic equivalence; we then say that $\mathcal{V}$ is *effectively meromorphic along Y*. I do not know if there exist coherent analytic sheaves on X^* that are meromorphic along Y but not effectively meromorphic along Y. p. 66

Let $\mathcal{V}$ be a coherent analytic sheaf on X^* that is meromorphic along Y; a *section* $v \in \mathrm{H}^0(X^*, \mathcal{V})$ is said to be *meromorphic along Y* if, locally on X, it is defined by a section of one of the pre-given extensions of $\mathcal{V}$. The information of the sheaf $j_*^{\mathrm{mero}}\mathcal{V}$ on X of sections of $\mathcal{V}$ which are meromorphic along Y is equivalent to the information of the meromorphic structure along Y of $\mathcal{V}$.

2.14. Suppose that X^* is reduced, and let $\mathcal{V}$ be a vector bundle on X^* which is meromorphic along Y. We will define an equivalence class of "norms" on $\mathcal{V}$. The "norms" in question will be continuous families of norms on the $\mathcal{V}_x$ (for $x \in X^*$). If v is a continuous section of $\mathcal{V}$, then $|v|$ is a positive function on X^*, which is zero at exactly the points where $v = 0$. Two norms $|v|_1$ and $|v|_2$ will be said to be equivalent if we have the following:

2.14.1. For any norm $\|x\|$ on X^*, and any compact subset K of X, there exist $A_1, A_2, N_1, N_2 > 0$ such that

$$|v|_1 \leqslant A_1(1 + \|x\|)^{N_1} |v|_2$$
$$|v|_2 \leqslant A_2(1 + \|x\|)^{N_2} |v|_1$$

on $K \cap X^*$.

2.15.1. If $\mathcal{V} = \mathcal{O}^n$, then we set $|v| = \sum |v_i|$.

2.15.2. Let $x \in Y$, and let $\mathcal{V}_1$ be a pre-given extension of $\mathcal{V}$ on a neighbourhood of x. Then there exists an open neighbourhood U of x, along with $\omega \colon \mathcal{V}_1 \to \mathcal{O}^n$, such that ω is a monomorphism on $U \cap X^*$. We set

$$|v|_\omega = |\omega(v)|$$

in $U \cap X^*$.

2.15.3. For x and $\mathcal{V}_1$ as in (2.15.1), there exists a neighbourhood U of x, and an epimorphism $\eta \colon \mathcal{O}^n \to \mathcal{V}_1$ on U. We set

$$|v|_\eta = \inf_{\eta(w)=v} |w|$$

in $U \cap X^*$. p. 67

2.15.4. We now compare (2.15.2) and (2.15.3). Consider U along with the meromorphic homomorphisms

$$\mathcal{O}^n \xrightarrow{\eta} \mathcal{V} \xrightarrow{\omega} \mathcal{O}^m$$

defined on $U \setminus Y$, with η an epimorphism and ω a monomorphism. The meromorphic homomorphism $\omega\eta$ is such that its kernel and image are locally direct factors. This still holds true in the scheme $\operatorname{Spec}(\mathcal{O}_{x,X}) \setminus Y_x$. We thus easily deduce that, for every holomorphic function f on U that vanishes on Y, there exists $N > 0$, and open neighbourhood $U_1 \subset U$ of x, and some $\alpha \colon \mathcal{O}^m \to \mathcal{O}^n$ such that

$$\eta\alpha\omega = f^N.$$

Let K be a compact subset of U_1. It is clear that there exists $M > 0$ such that

$$|\omega\eta(v)| \leqslant C^{te}(1 + \|x\|)^M |v|$$
$$|\alpha(v)| \leqslant C^{te}(1 + \|x\|)^M |v|$$

on K, and thus

$$|v|_\omega \leqslant C^{te}(1 + \|x\|)^M |v|_\eta \tag{1}$$
$$|v|_\eta \leqslant C^{te}(1 + \|x\|)^M |f^N| |v|_\omega \tag{2}$$

on K.

We can apply (2) to a finite family of functions f_i that generates an ideal of definition of Y.

By (2.4) and (2.5), there exists M' such that

$$\sum_i |f_i^N| \leqslant C^{te}(1 + \|x\|)^{M'} \tag{3}$$

on K, and it thus follows that, in a small enough neighbourhood of x, $|v|_\omega$ and $|v|_\eta$ are equivalent, in the sense of (2.14.1). The equivalence (2.14.1) is of a local nature on X. We can thus prove, by using a partition of unity, the following proposition:

Proposition 2.16. *Under the hypotheses of (2.14), there exists exactly one equivalence class (2.14.1) of norms on $\mathcal{V}$ which are locally equivalent (in the sense of (2.14.1)) to the norms (2.15.2) and (2.15.3).*

We say that the norms whose existence is guaranteed by (2.16) are *moderate*. p. 68

Definition 2.17. Under the hypotheses of (2.14), let $|v|$ be a moderate norm on $\mathscr{V}$, $\pi : \widetilde{X}^* \to X^*$ a covering of X^*, and v a continuous section of $\pi^*\mathscr{V}$. We say that v is *of moderate growth along* Y if $|v|$ is of moderate growth along Y (2.10).

In the particular case where π is the identity, we can instead use Definition 2.6. This is the case in the following well-known proposition, which shows that the knowledge of moderate norms on $\mathscr{V}$ is equivalent to the knowledge of the meromorphic structure of $\mathscr{V}$ along Y.

Proposition 2.18. *Under the hypotheses of (2.14), for a holomorphic section v of $\mathscr{V}$ over X^* to be meromorphic along Y, it is necessary and sufficient for it to be of moderate growth along Y.*

Proof. The claim is local on X, and we can reduce, by (2.16) and (2.15.2) to the classical case where $\mathscr{V} = \mathscr{O}$. □

Proposition 2.19. *Consider a commutative diagram of separated analytic spaces*

$$\begin{array}{ccccc} X_1^* & \hookrightarrow & X_1 & \hookleftarrow & Y_1 \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ X_2^* & \hookrightarrow & X_2 & \hookleftarrow & Y_2 \end{array}$$

where Y_i is closed in X_i, and $X_i^* = X_i \setminus Y_i$, and thus $Y_1 = f^{-1}(Y_2)$.

Consider the hypotheses:

(a) *There exists a subset K of X_1 that is proper over X_2 and such that $f(K_1) \supset \overline{X_2^*}$.*

(b) *f is proper and induces an isomorphism $X_1^* \xrightarrow{\sim} X_2^*$.*

It is clear that (b) $\Longrightarrow$ (a). We have

($\mathrm{i_a}$) *If $\|x\|$ is a norm on X_2^* adapted to X_2, then $\|f(x)\|$ is a norm on X_1^* adapted to X_1.* p. 69

($\mathrm{i_b}$) *Conversely, if* (a) *is satisfied, and if $\|x\|$ is a function on X_2^* such that $\|f(x)\|$ is a norm on X_1^*, then $\|x\|$ is a norm on X_2^*.*

($\mathrm{i_c}$) *In particular, if* (b) *is satisfied, then the norms on $X_1^* = X_2^*$ adapted to X_1 or X_2 agree.*

Let $\pi_2 : \widetilde{X_2}^* \to X_2^*$ be a covering, and $\pi_1 : \widetilde{X_1}^* \to X_1^*$ its inverse image over X_1^*.

($\mathrm{ii_a}$) *If P is a subset of $\widetilde{X_1}^*$ which is vertical along Y_1, then $f(P)$ is vertical along Y_2.*

($\mathrm{ii_b}$) *Conversely, if* (a) *is satisfied, then every vertical subset of $\widetilde{X_2}^*$ is the image of a vertical subset of $\widetilde{X_1}^*$.*

($\mathrm{ii_c}$) *In particular, if* (b) *is satisfied, then the subsets of $\widetilde{X_1}^* = \widetilde{X_2}^*$ which are vertical along Y_1 or Y_2 agree.*

Let $\mathcal{V}_2$ be a vector bundle on X_2^* that is meromorphic along Y_2, and let $\mathcal{V}_1$ be its inverse image. The inverse images of the pre-given extensions of $\mathcal{V}_2$ define a meromorphic structure along Y_1 on $\mathcal{V}_1$, and we have

(iii$_a$) *The inverse image of a moderate norm on $\mathcal{V}_2$ is a moderate norm on $\mathcal{V}_1$.*

(iii$_b$) *Conversely, if* (a) *is satisfied, then a norm on $\mathcal{V}_2$ is moderate if its inverse image is.*

(iii$_c$) *In particular, if* (b) *is satisfied, then the moderate norms on $\mathcal{V}_1 = \mathcal{V}_2$ adapted to X_1 or X_2 agree.*

Proof. We trivially have that (i$_a$) + (i$_b$) $\Longrightarrow$ (i$_c$), (ii$_a$) + (ii$_b$) $\Longrightarrow$ (ii$_c$), (iii$_a$) + (iii$_b$) $\Longrightarrow$ (iii$_c$), and almost trivially that (i$_a$) $\Longrightarrow$ (i$_b$) and (i$_a$) + (iii$_a$) $\Longrightarrow$ (iii$_b$).

If Y_2 is defined by equations $f_i = 0$, then Y_1 is defined by the inverse image of the f_i, and (i$_a$) follows from Definition 2.5 (along with (2.4.1)).

By (2.15.2) and (2.16), we can reduce to proving (iii$_a$) in the trivial case where $\mathcal{V} = \mathcal{O}$. p. 70

Finally, if $\mathcal{T}_2$ is a semi-analytic triangulation of the pair (X_2, Y_2), then the $f^{-1}(\sigma)$ (for $\sigma \in \mathcal{T}_2$) form a locally finite set of semi-analytic subsets of X_1, and, by 2.8.2, there exists, locally on X_1, a semi-analytic triangulation $\mathcal{T}_1$ of the pair (X_1, Y_1) such that

$$\forall \sigma \in \mathcal{T}_1 \quad \exists \tau \in \mathcal{T}_2 \quad \text{such that } f(\sigma) \subset \tau.$$

Claims (ii$_a$) and (ii$_b$) thus follow immediately. □

Remarks 2.20.

(i) Let $X = D^{n+m}$ and $X^* = (D^*)^n \times D^m$; then Y is a normal crossing divisor in X. On the universal cover $\widetilde{X}^*$ of X^*, the functions $\arg(z_i)$ (for $1 \leqslant i \leqslant n$) are defined. It is clear that a subset P of $\widetilde{X}^*$ is vertical along Y if and only if its image in X is relatively compact and if the functions $\arg(z_i)$ (for $1 \leqslant i \leqslant n$) are bounded on P.

(ii) Hironaka's resolution of singularities, along with (ii$_c$) of (2.19), allows us, in the general case, to make explicit the idea of vertical subsets, starting from the particular case of (i).

2.21. Proposition 2.19 allows us to pursue the programme of (2.1). So let X^* be a separated scheme of finite type over $\mathbb{C}$, and X a proper scheme over $\mathbb{C}$ that contains X^* as a Zariski open subset. If $\mathcal{F}$ is a coherent algebraic sheaf on X^*, then we know (EGA I, 9.4.7) that $\mathcal{F}$ can be extended to a coherent algebraic sheaf $\mathcal{F}_1$ on X. The various sheaves $\mathcal{F}_1^{\mathrm{an}}$ define on $\mathcal{F}^{\mathrm{an}}$ (effectively) the same meromorphic structure along $Y = X \setminus X^*$.

We further immediately deduce, from [GAGA], the following:

Proposition 2.22. *Under the hypotheses of (2.21), the functor $\mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$ induces an equivalence between the category of coherent algebraic sheaves on X^* and the category of coherent analytic sheaves on $(X^*)^{\mathrm{an}}$ that are effectively meromorphic along Y.*

Definition 2.23. Let X^* be a separated scheme of finite type over $\mathbb{C}$, and let X be as in (2.21). Let $\pi\colon \widetilde{X}^* \to (X^*)^{\mathrm{an}}$ be a covering of $(X^*)^{\mathrm{an}}$, and $\mathcal{V}$ an algebraic vector bundle on X^*. p. 71

(i) A *norm on* X^* is defined to be a norm on $(X^*)^{\mathrm{an}}$ that is adapted to X^{an} (2.5).

(ii) A *vertical subset of* $\widetilde{X}^*$ is defined to be a subset of $\widetilde{X}^*$ that is vertical along $Y = X \setminus X^*$ (2.9).

(iii) A *moderate norm on* $\mathcal{V}$ is defined to be a moderate norm on $\mathcal{V}^{\mathrm{an}}$ with respect to the meromorphic structure at infinity of $\mathcal{V}^{\mathrm{an}}$ (2.16).

(iv) A continuous section v of $\pi^*\mathcal{V}$ is said to be of *moderate growth* if it is of moderate growth along Y (2.17).

By (2.19), these definitions do not depend on the choice of compactification X of X^*. We can also immediately deduce, from (2.18) and [GAGA] (as in (2.22)), the following:

Proposition 2.24. *Let X be a separated scheme that is reduced and of finite type over $\mathbb{C}$, and let $\mathcal{V}$ be an algebraic vector bundle on X. Then a holomorphic section v of $\mathcal{V}^{\mathrm{an}}$ is algebraic if and only if it is of moderate growth.*

Problem 2.25. Let $X = G/K$ be a Hermitian symmetric domain (with G a real Lie group, and K a compact maximal subgroup), and Γ an arithmetic subgroup of G. The quotient $\Gamma\backslash G/K$ is then naturally a quasi-projective algebraic variety [2]. Is it true that a subset P of G/K is vertical (2.23) if and only if it is contained in the union of a finite number of Siegel domains?

II.3 Logarithmic poles

This section brings together some constructions that are "local at infinity **!TO-DO!** " of which we will later have use. *p. 72*

Definition 3.1. Let Y be a normal crossing divisor in a complex-analytic variety X, and let j be the inclusion of $X^* = X \setminus Y$ into X. We define the *logarithmic de Rham complex* of X along Y to be the smallest sub-complex $\Omega^\bullet_X\langle Y\rangle$ of $j_*\Omega^\bullet_{X^*}$ that containing $\Omega^\bullet_X$ that is stable under the exterior product, and such that $\mathrm{d}f/f$ is a local section of $\Omega^1_X\langle Y\rangle$ for all local sections f of $j_*\Omega^\bullet_X$ that are meromorphic along Y.

A section of $j_*\Omega^p_{X^*}$ is said to present a *logarithmic pole* along Y if it is a section of $\Omega^p_X\langle Y\rangle$.

Proposition 3.2. *Under the hypotheses of (3.1),*

(i) For a section α of $j_*\Omega^p_{X^*}$ to present a logarithmic pole along Y, it is necessary and sufficient for α and $\mathrm{d}\alpha$ to present at worst simple poles along Y.

(ii) The sheaf $\Omega^1_X\langle Y\rangle$ is locally free, and

$$\Omega^p_X\langle Y\rangle = \bigwedge^p \Omega^1_X\langle Y\rangle.$$

(iii) If the pair (X,Y) *is a product* $(X,Y)=(X_1,Y_1)\times(X_2,Y_2)$*, i.e. if*

$$X=X_1\times X_2 \quad \textit{and} \quad Y=X_1\times Y_2\cup X_2\times Y_1,$$

then the isomorphism between $\Omega^\bullet_{X^*}$ *and the external tensor product* $\Omega^\bullet_{X_1^*}\boxtimes\Omega^\bullet_{X_2^*}$ *(defined by* $\mathrm{pr}_1^*\Omega^\bullet_{X_1^*}\otimes\mathrm{pr}_2^*\Omega^\bullet_{X_2^*}$*) induces an isomorphism*

$$\Omega^\bullet_{X_1}\langle Y_1\rangle\boxtimes\Omega^\bullet_{X_2}\langle Y_2\rangle \xrightarrow{\sim} \Omega^\bullet_X\langle Y\rangle.$$

(iv) Let Y_i *be a normal crossing divisor in* X_i *(for* $i=1,2$*), and* $f\colon X_1\to X_2$ *a morphism such that* $f^{-1}(Y_2)=Y_1$*. Then the morphism* $f^*\colon f^*((j_2)_*\Omega^\bullet_{X_2^*})\to(j_1)_*\Omega^\bullet_{X_1^*}$ *induces an "inverse image" morphism*

$$f^*\colon f^*\Omega^\bullet_{X_2}\langle Y_2\rangle\to\Omega^1_{X_1}\langle Y_1\rangle.$$

p. 73

Proof. Claim (iv) is trivial, given the definition. Let D be the open unit disc, and $D^*=D\setminus\{0\}$. To prove (i), (ii), and (iii), we can assume that X is the polydisc D^n, and that $X^*=(D^*)^k\times D^{n-k}$, and $Y=\bigcup_{1\leqslant i\leqslant k}Y_i$, where $Y_i=\mathrm{pr}_i^{-1}(0)$. Under these hypotheses, we have

Lemma 3.2.1. *The sheaf* $\Omega^1_X\langle Y\rangle$ *is free, with basis* $(\mathrm{d}z_i/z_i)_{1\leqslant i\leqslant k}\cup(\mathrm{d}z_j)_{k<j\leqslant n}$*.*

Proof. Indeed, every section of $j_*\mathcal{O}^*_{X^*}$ that is meromorphic along Y can be written locally as $f=g\cdot\prod_{i=1}^k z_i^{k_i}$ with g invertible, and

$$\mathrm{d}f/f=\mathrm{d}g/g+\sum_{i=1}^k k_i\mathrm{d}z_i/z_i$$

is a linear combination of the proposed basis vectors, which are clearly independent. □

From this lemma, we immediately deduce (ii), (iii), and the necessity of the condition in (i).

Let α be a section of $j_*\Omega^p_{X^*}$ satisfying the condition of (i). To prove that α is a section of $\Omega^p_{X^*}\langle Y\rangle$, it suffices (since this sheaf is locally free) to prove this outside of a set of complex codimension $\geqslant 2$. This allows us to suppose that the hypotheses of (3.2.1) are satisfied, with $k=1$. The form α can thus be written in the form $\alpha=\alpha_1+\alpha_2\wedge\mathrm{d}z_1/z_1$ in exactly one way, with the forms α_1 and α_2 being such that they do not contain any $\mathrm{d}z_1$ term.

The hypotheses imply that

- α_2 is holomorphic;
- $z_1\alpha_1$ is holomorphic; and
- $z_1\mathrm{d}\alpha=z_1\mathrm{d}\alpha_1+\mathrm{d}\alpha_2\wedge\mathrm{d}z_1$ is holomorphic.

From this,

$$\mathrm{d}z_1\wedge\alpha_1=\mathrm{d}(z_1\wedge\alpha_1)+z_1\mathrm{d}\alpha-\mathrm{d}\alpha_2\wedge\mathrm{d}z_1$$

is holomorphic, and thus so too is α_1, which proves (i). □

Variants 3.3.

3.3.1. Let $f: X \to S$ be a smooth morphism of schemes of characteristic 0, or a smooth morphism of analytic spaces, and let Y be a relative normal crossing divisor in X. Definition 3.1 still makes sense, and defines a sub-complex $\Omega^\bullet_{X/S}\langle Y\rangle$ of $j_*\Omega^\bullet_{X^*/S}$ (where j is the inclusion $j: X^* = X \setminus Y \to X$). Proposition 3.2 still holds true, mutatis mutandis. Forming the complex $\Omega^\bullet_{X/S}\langle Y\rangle$ is compatible with any base change, and with étale localisation on X.

3.3.2. Let $f: X \to S$ be a morphism of smooth analytic spaces, 0 a point of S, and Y a normal crossing divisor in X. Let $S^* = S \setminus \{0\}$, and $X^* = X \setminus Y$, and let j be the inclusion of X^* into X. Suppose that

(a) $\dim(S) = 1$;

(b) $f|f^{-1}(S^*)$ is smooth, and $Y \cap f^{-1}(S^*)$ is a relative normal crossing divisor in $f^{-1}(S^*)$; and

(c) $Y \supset f^{-1}(0)$.

We can then define the complex $\Omega^\bullet_{X/S}\langle Y\rangle$ as the image of $j_*\Omega^\bullet_{X^*}$ in $\Omega^\bullet_X\langle Y\rangle$.

Locally, close to 0 and $f^{-1}(0)$, we can find coordinate systems $(z_i)_{0\leqslant i\leqslant n}$ on X, and z on S, such that $z(0) = 0$, such that $z \circ f = \prod_{i=0}^k z_i^{e_i}$ (for $k \leqslant n$ and $e_i > 0$), and such that Y can be described by the equation $\prod_{i=0}^l z_i = 0$ (for $k \leqslant l \leqslant n$). In such a coordinate system, the sheaf $\Omega^1_{X/S}\langle Y\rangle$ is free, with basis given by $(\mathrm{d}z_i/z_i)_{1\leqslant i\leqslant l} \cup (\mathrm{d}z_j)_{l<j\leqslant n}$. In $\Omega^1_{X/S}\langle Y\rangle$, we have the equation

$$\mathrm{d}f/f = \sum_{i=0}^{k} e_i \mathrm{d}z_i/z_i = 0.$$

We thus deduce that $\Omega^1_{X/S}\langle Y\rangle$ is *locally free*, that

$$\bigwedge^p \Omega^1_{X/S}\langle Y\rangle \xrightarrow{\sim} \Omega^p_{X/S}\langle Y\rangle, \tag{3.3.2.1}$$

and that the sequence p. 75

$$0 \to f^*\Omega^1_S\langle 0\rangle \xrightarrow{f^*} \Omega^1_X\langle Y\rangle \to \Omega^1_{X/S}\langle Y\rangle \to 0 \tag{3.3.2.2}$$

is exact and *locally split*. This will play a key role in (II.7), in the following form:

Lemma 3.3.2.3. *Every vector field v_0 on S that vanishes at 0 can be locally lifted to a vector field v on X satisfying*

$$\left\langle v, \Omega^1_X\langle Y\rangle\right\rangle \subset \mathscr{O}_X.$$

Indeed, the transpose of the direct monomorphism f^* in (3.3.2.2) is an epimorphism.

3.3.3. The reader can translate (3.3.2) into the setting of a morphism $f: X \to S$ of schemes of finite type over $\mathbb{C}$ that satisfies the conditions analogous to (3.3.2.a,b,c).

3.4. Let Y be a normal crossing divisor in S. Locally on X, we can write Y as a sum of smooth divisors Y_i. We denote by Y^p (resp. $\widetilde{Y}^p$) the union (resp. disjoint sum) of the

p-fold intersections (?? **!TO-DO!** ??) of the Y_i; the Y^p, thus locally defined, glue to give a subspace Y^p of X, and the $\widetilde{Y}^p$ glue to give the normalised variety of Y^p. We have $\widetilde{Y}^0 = Y^0 = X$, and we set $\widetilde{Y} = Y^1$. Let $a\colon Y^p \to X$ be the projection.

If, to each point $y \in \widetilde{Y}^p$, we associate the set of **!TO-DO!** , then we define a local system E_p on $\widetilde{Y}^p$ of sets with p elements.

Denote by ϵ^p the rank-1 local system

$$\epsilon^p = \bigwedge^p \underline{\mathbb{C}}^{E_p}$$

on $\widetilde{Y}^p$. We have that $(\epsilon^p)^{\otimes 2} \simeq \underline{\mathbb{C}}$. If Y is the sum of smooth divisors $(Y_i)_{i\in I}$, then the choice of a total order on I trivialises the ϵ^p.

3.5. Denote by $W_n(\Omega_X^\bullet\langle Y\rangle)$ the smallest sub-$\mathcal{O}$-module of $\Omega_X^\bullet\langle Y\rangle$ that is stable under the exterior product with the local sections of $\Omega_X^\bullet$ and that contains the products

$$\mathrm{d}f_1/f_1 \wedge \ldots \wedge \mathrm{d}f_k/f_k$$

for $k \leqslant n$, and local sections f_i of $j_*\mathcal{O}_{X^*}^*$ that are meromorphic along Y. Then the W_n form an *increasing* filtration of $\Omega_X^\bullet\langle Y\rangle$ by sub-complexes, called the *weight filtration*. We have that p. 76

$$W_n(\Omega_X^\bullet\langle Y\rangle) \wedge W_m(\Omega_X^\bullet\langle Y\rangle) \subset W_{n+m}(\Omega_X^\bullet\langle Y\rangle). \tag{3.5.1}$$

Locally on X, we can write Y as a finite sum of smooth divisors $(Y_i)_{i\in I}$ defined by equations $t_i = 0$. Let q be an injection of $[1,n]$ into I, let $e(q)$ be the corresponding section $e_{q(1)} \wedge \ldots \wedge e_{q(n)}$ of ϵ^n on the component $Y_q = \bigcap_{1\leqslant i\leqslant n} Y_{q(i)}$ of $\widetilde{Y}^n$, and let $a_q\colon Y_q \to X$ be the projection.

The map ρ_0 from Ω_X^p to $W^n/W^{n-1}(\Omega_X^{p+n}\langle Y\rangle)$ given by

$$\alpha \longmapsto \mathrm{d}t_{(q_1)}/t_{q(1)} \wedge \ldots \wedge \mathrm{d}t_{q(n)}/t_{q(n)} \wedge \alpha \tag{3.5.2}$$

does not depend on the choice of the t_i, since, if t_i' are a different choice, then the $\mathrm{d}t_i/t_i - \mathrm{d}t_i'/t_i' = \mathrm{d}(t_i/t_i')/(t_i/t_i')$ are holomorphic, and $\rho_0(\alpha) - \rho_0'(\alpha) \in W^{n-1}(\Omega_X^{p+n}\langle Y\rangle)$. Similarly, $\rho_0(t_{q(i)}\cdot\beta) = 0$, and $\rho_0(\mathrm{d}t_{q(i)} \wedge \beta) = 0$, and so ρ_0 factors through

$$\rho_1\colon (a_q)_*\Omega_{Y_q}^p \to W^n/W^{n-1}(\Omega_X^{p+n}\langle Y\rangle).$$

The trivialisation $e(q)$ of $\epsilon^n|Y_q$ identifies ρ_1 with

$$\rho_2\colon (a_q)_*\Omega_{Y_q}^\bullet(\epsilon^n)[-n] \to \mathrm{Gr}_n^W(\Omega_X^\bullet\langle Y\rangle).$$

Finally, the sum of the morphisms ρ_2 for the different q defines a morphism of complexes

$$\rho\colon a_*\Omega_{\widetilde{Y}^n}^\bullet(\epsilon)[-n] \to \mathrm{Gr}_n^W(\Omega_X^\bullet\langle Y\rangle). \tag{3.5.3}$$

This morphism, defined locally by (3.5.2), glues to give a morphism of complexes on X.

Proposition 3.6. *The morphisms (3.5.3) are isomorphisms.*

Proof. If the pair (X,Y) is a product $(X,Y) = (X_1,Y_1)\times(X_2,Y_2)$, i.e. if

$$X = X_1 \times X_2 \quad \text{and} \quad Y = X_1 \times Y_2 \cup X_2 \times Y_1,$$

then the weight filtration on $\Omega^\bullet_X\langle Y\rangle$ is the external tensor product, via (3.2.iii), of the weight filtrations on the $\Omega^\bullet_{X_i}\langle Y_i\rangle$. We thus have p. 77

$$\mathrm{Gr}^W(\Omega^\bullet_{X_1}\langle Y\rangle)\boxtimes\mathrm{Gr}^W(\Omega^\bullet_{X_2}\langle Y_2\rangle)\xrightarrow{\sim}\mathrm{Gr}^W(\Omega^\bullet_X\langle Y\rangle). \tag{3.6.1}$$

The isomorphisms

$$\widetilde{Y}^n = \coprod_{p+q=n}\widetilde{Y}^p\boxtimes\widetilde{Y}^q$$
$$\epsilon^n = \coprod_{p+q=n}\epsilon^p\boxtimes\epsilon^q$$

induce an isomorphism

$$\sum_p a_*\Omega^\bullet_{\widetilde{Y}^p}(\epsilon^p)[-p]\boxtimes\sum_q a_*\Omega^\bullet_{\widetilde{Y}^q}(\epsilon^q)[-q]\xrightarrow{\sim}\sum_n a_*\Omega^\bullet_{\widetilde{Y}^n}(\epsilon^n)[-n]. \tag{3.6.2}$$

Further, via (3.6.1) and (3.6.2), we have

$$\rho_1\boxtimes\rho_2 = \rho. \tag{3.6.3}$$

For ρ to be an isomorphism, it is thus sufficient that the ρ_i be isomorphisms. Since the problem is local on X, this allows us to restrict to the trivial case where $\dim(X) = 1$. □

3.7. The isomorphism inverse to ρ is called the *Poincaré residue*

$$\mathrm{Res}\colon \mathrm{Gr}^W_n(\Omega^p_X\langle Y\rangle)\to\Omega^p_{\widetilde{Y}^n}(\epsilon^n)[-n]. \tag{3.7.1}$$

We will only need the case where $p=1$, which gives

$$\mathrm{Res}\colon \Omega^1_X\langle Y\rangle\to\mathcal{O}_{\widetilde{Y}}. \tag{3.7.2}$$

If $\mathcal{V}$ is a vector bundle on X, then the morphism (3.7.2) extends by linearity to

$$\mathrm{Res}\colon \Omega^1_X\langle Y\rangle(\mathcal{V})\to\mathcal{O}_{\widetilde{Y}}\otimes\mathcal{V}. \tag{3.7.3}$$

For each smooth component Y_i of Y, this gives p. 78

$$\mathrm{Res}_{Y_i}\colon \Omega^1_X\langle Y\rangle(\mathcal{V})\to\mathcal{V}|Y_i. \tag{3.7.4}$$

3.8. Under the hypotheses of (3.1), let $\mathcal{V}_0$ be a vector bundle on $X*$, endowed with an integrable connection ∇. Suppose that $\mathcal{V}_0$ is given as the restriction to X^* of a vector bundle $\mathcal{V}$ on X. Locally on X, the choice of a basis e of $\mathcal{V}$ allows us to define the "connection matrix"

$$\Gamma\in j_*\Omega^1_{X^*}(\underline{\mathrm{End}}(\mathcal{V})). \tag{3.8.1}$$

A change of basis $e\mapsto e'$ modifies Γ by addition of a section of $\Omega^1_X(\underline{\mathrm{End}}(\mathcal{V}))$ (I.3.1.3). Thus the "polar part of Γ"

$$\dot{\Gamma}\in j_*\Omega^1_{X^*}\big(\underline{\mathrm{End}}(\mathcal{V}_0)\big)/\Omega^1_X\big(\underline{\mathrm{End}}(\mathcal{V})\big) \tag{3.8.2}$$

depends only on $\mathcal{V}$ and on ∇. We say that *the connection* ∇ *has at worst a logarithmic pole along* Y if, in every local basis of $\mathcal{V}$, the connection forms present at worst a logarithmic pole along Y. In this case, *the residue of the connection* Γ along a local component Y_i of Y is defined (3.7.4)

$$\mathrm{Res}_{Y_i}(\Gamma) \in \underline{\mathrm{End}}(\mathcal{V}|Y_i) \tag{3.8.3}$$

and it depends only on $\mathcal{V}$ and on ∇. More globally, if $i\colon \widetilde{Y} \to X$ is the projection of the normalisation of Y onto X, then the residue of the connection along Y is an endomorphism of $i^*\mathcal{V}$

$$\mathrm{Res}_Y(\Gamma) \in \underline{\mathrm{End}}(i^*\mathcal{V}). \tag{3.8.4}$$

3.9. We place ourselves under the hypotheses of (3.8), and make the following additional hypotheses:

a) Y is the sum of smooth divisors $(Y_i)_{1\leqslant i\leqslant n}$ (which is always the case locally). For $P \subset [1,n]$, we set $Y_P = \bigcap_{i\in P} Y_i$ and $Y'_P = Y_P \setminus \bigcup_{i\notin P} Y_i$.

b) The connection on $\mathcal{V}$ has at worst a logarithmic pole along Y.

The dual of the vector bundle $\Omega^1_X\langle Y\rangle$ is the bundle $T^1_X\langle -Y\rangle$ of vector fields v on X that satisfy p. 79

3.9.1. For $P \subset [1,n]$, $v|Y_P$ is tangent to Y_P.

If a vector field v satisfies (3.9.1), and if g is a section of $\mathcal{V}$, then $\nabla_v(g)$ is again a regular section of $\mathcal{V}$. Its restriction to Y_P depends only on $g|Y_P$ and on the image of v in $T^1_X\langle -Y\rangle \otimes \mathcal{O}_{Y_P}$. If s is a local section of the evident epimorphism from $T^1_X\langle -Y\rangle \otimes \mathcal{O}_{Y_P}$ to its image in the tangent bundle to Y_P, then $\nabla_{s(v)}(g)$ defines a connection ${}_s\nabla$ on $\mathcal{V}|Y'_P$. There is a Lie bracket defined, by passing to the quotient, on $T^1_X\langle -Y\rangle \otimes \mathcal{O}_{Y_P}$. The connection ${}_s\nabla$ is integrable if s commutes with the bracket; it presents at worst a logarithmic pole along $Y_P \cap \bigcup_{i\notin P} Y_i$.

An easy calculation shows the following:

Proposition 3.10. *Under the previous hypotheses, and with the above notation,*

(i) $[\mathrm{Res}_{Y_i}(\Gamma), \mathrm{Res}_{Y_j}(\Gamma)] = 0$ *on* $Y_i \cap Y_j$*; and*

(ii) if $i \in P$*, then* ${}_s\nabla\,\mathrm{Res}_{Y_i}(\Gamma) = 0$ *on* Y'_P.

We deduce from (3.10.ii), for $P = \{i\}$, that the characteristic polynomial of $\mathrm{Res}_{Y_i}(\Gamma)$ is constant on Y_i.

We can also deduce (3.10) from the following proposition, which can be proven similarly to (1.17):

Proposition 3.11. *Let* $\mathcal{V}$ *be a vector bundle on* $X = D^n$*, let* $Y = \{0\} \times D^{n-1}$*, and* $X^* = X \setminus Y$*, and let* Γ *be an integrable connection on* $\mathcal{V}|X^*$ *presenting a logarithmic pole along* Y*. Let* T *be the monodromy transformation of* $\mathcal{V}|X^*$ *defined by the positive generator of* $\pi_1(X^*) \cong \pi_1(D^*) \cong \mathbb{Z}$ *(cf. (1.15)). Then the horizontal automorphism* T *of* $\mathcal{V}|X^*$ *can be extended to an automorphism of* $\mathcal{V}$*, again denoted by* T*, and*

$$T|Y = \exp(-2\pi i\,\mathrm{Res}_Y(\Gamma)).$$

3.12. Let X be a complex-analytic variety, Y a normal crossing divisor in X, and $j\colon X^* = X \setminus Y \to X$ the inclusion. For a vector bundle $\mathcal{V}$ on X, we denote by $j_*^{\mathrm{m}} j^* \mathcal{V}$ the sheaf of sections of $\mathcal{V}$ over X^* that are meromorphic along Y. *p. 80*

Locally on X, Y is the union of smooth divisors Y_i, and we define the *pole-order filtration* P of $j_*^{\mathrm{m}} j^* \mathcal{O} = j_*^{\mathrm{m}} \mathcal{O}_{X^*}$ by the equation

$$P^k(j_*^{\mathrm{m}} j^* \mathcal{O}) = \sum_{\underline{n} \in A_k} \mathcal{O}\big(\textstyle\sum (n_i + 1) Y_i\big) \tag{3.12.1}$$

where

$$A_k = \big\{(n_i) \mid \textstyle\sum_i n_i \leqslant -k \text{ and } n_i \geqslant 0 \text{ for all } i\big\}.$$

This construction can be made global, and endows $j_*^{\mathrm{m}} \mathcal{O}_{X^*}$ with an exhaustive filtration such that $P^k = 0$ for $k > 0$.

Let $\mathcal{V}$ be a vector bundle on X, and Γ an integrable connection on $\mathcal{V}|X^*$ presenting a logarithmic pole along Y. We define a filtration P, again called the *pole-order filtration*, of the complex $j_*^{\mathrm{m}} j^* \Omega_X^\bullet(\mathcal{V}) = j_*^{\mathrm{m}} \Omega_{X^*}^\bullet(\mathcal{V})$ by

$$P^k\big(j_*^{\mathrm{m}} \Omega_{X^*}^p(\mathcal{V})\big) = P^{k-p}(j_*^{\mathrm{m}} \mathcal{O}_{X^*}) \otimes \Omega_X^p \otimes \mathcal{V}. \tag{3.12.2}$$

From the fact that Γ presents at worst logarithmic poles along Y, we deduce that

a) the filtration P in (3.12.2) is compatible with the differentials; and

b) $\Omega_X^\bullet\langle Y\rangle(\mathcal{V})$ is a sub-complex of $j_*^{\mathrm{m}} \Omega_{X^*}^\bullet(\mathcal{V})$.

Furthermore,

c) the operators d are $\mathcal{O}_X$-linear on the complexes $\mathrm{Gr}_P^n(j_*^{\mathrm{m}} \Omega_{X^*}^\bullet(\mathcal{V}))$; and

d) the filtration P induces the *Hodge filtration* F on $\Omega_X^\bullet\langle Y\rangle(\mathcal{V})$ by the stupid truncations $\sigma_{\geqslant p}$, whence we have a morphism of filtered complexes

$$\big(\Omega_X^\bullet\langle Y\rangle(\mathcal{V}), F\big) \to \big(j_*^{\mathrm{m}} \Omega_{X^*}^\bullet(\mathcal{V}), P\big). \tag{3.12.3}$$

Proposition 3.13. *With the hypotheses and notation of (3.12), if the residues of the connection Γ along all the local components of Y do not admit any strictly positive integer as an eigenvalue, then*

(i) the morphism of complexes (3.12.3) is a quasi-isomorphism; and

(ii) more precisely, it induces a quasi-isomorphism *p. 81*

$$\mathrm{Gr}_F\big(\Omega_X^\bullet\langle Y\rangle(\mathcal{V})\big) \to \mathrm{Gr}_P\big(j_*^{\mathrm{m}} \Omega_{X^*}^\bullet(\mathcal{V})\big). \tag{3.13.1}$$

Proof. It suffices to prove (ii), which also implies that, for each p, $\mathrm{Gr}_P^p(j_*^{\mathrm{m}} \Omega_{X^*}^\bullet(\mathcal{V}))[p]$ is a resolution of $\Omega_X^p\langle Y\rangle(\mathcal{V})$.

First reduction (Extensions). If $\mathcal{V}$ is an extension of bundles with connections satisfying (3.13), as in

$$0 \to \mathcal{V}' \to \mathcal{V} \to \mathcal{V}'' \to 0$$

then the lines of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Gr}_F\,\Omega^\bullet_X\langle Y\rangle(\mathcal{V}') & \longrightarrow & \mathrm{Gr}_F\,\Omega^\bullet_X\langle Y\rangle(\mathcal{V}) & \longrightarrow & \mathrm{Gr}_F\,\Omega^\bullet_X\langle Y\rangle(\mathcal{V}'') & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Gr}_P\, j^{\mathrm{m}}_*\Omega^\bullet_{X^*}(\mathcal{V}') & \longrightarrow & \mathrm{Gr}_P\, j^{\mathrm{m}}_*\Omega^\bullet_{X^*}(\mathcal{V}) & \longrightarrow & \mathrm{Gr}_P\, j^{\mathrm{m}}_*\Omega^\bullet_{X^*}(\mathcal{V}'') & \longrightarrow & 0 \end{array}$$

are exact. For (3.13.1) to be a quasi-isomorphism, it thus suffices to prove the analogous claim for $\mathcal{V}'$ and $\mathcal{V}''$.

Second reduction (Products). Suppose that (X,Y) is the product of (X_1,Y_1) and (X_2,Y_2), and that $\mathcal{V}$ is the external tensor product $\mathcal{V} = \mathcal{V}_1 \boxtimes \mathcal{V}_2$ of bundles $\mathcal{V}_i$ with connections satisfying the hypotheses of (3.13) on (X_i,Y_i).

The isomorphism (3.2.iii) identifies the Hodge filtration of $\Omega^\bullet_X\langle Y\rangle$ with the external tensor product of the Hodge filtrations of the $\Omega^\bullet_{X_i}\langle Y_i\rangle$, whence the evident isomorphism

$$\mathrm{Gr}_F\left(\Omega^\bullet_{X_1}\langle Y_1\rangle(\mathcal{V}_1)\right) \boxtimes \mathrm{Gr}_F\left(\Omega^\bullet_{X_2}\langle Y_2\rangle(\mathcal{V}_2)\right) \xrightarrow{\sim} \mathrm{Gr}_F\left(\Omega^\bullet_X\langle Y\rangle(\mathcal{V})\right). \tag{3.13.2}$$

The evident isomorphism

$$j^{\mathrm{m}}_*\Omega^\bullet_{X_1^*} \boxtimes j^{\mathrm{m}}_*\Omega^\bullet_{X_2^*} \xrightarrow{\sim} j^{\mathrm{m}}_*\Omega^\bullet_{X^*}$$

identifies the filtration P of $j^{\mathrm{m}}_*\Omega^\bullet_{X^*}$ with the external tensor product of the filtrations p. 82
P of the $j^{\mathrm{m}}_*\Omega^\bullet_{X_i^*}$. We thus have

$$\mathrm{Gr}_P\left(j^{\mathrm{m}}_*\Omega^\bullet_{X_1^*}(\mathcal{V}_1)\right) \boxtimes \mathrm{Gr}_P\left(j^{\mathrm{m}}_*\Omega^\bullet_{X_2^*}(\mathcal{V}_2)\right) \xrightarrow{\sim} \mathrm{Gr}_P\left(j^{\mathrm{m}}_*\Omega^\bullet_{X^*}(\mathcal{V})\right). \tag{3.13.3}$$

The morphism (3.13.1) can be identified, via (3.13.2) and (3.13.3), with the external tensor product of the analogous morphisms for $\mathcal{V}_1$ and $\mathcal{V}_2$. Since the complexes in question have $\mathcal{O}_X$-linear differentials (3.12.d), to prove that is a quasi-isomorphism, it suffices to prove the analogous claim for $\mathcal{V}_1$ and $\mathcal{V}_2$.

Case of constant coefficients. We first prove (3.13.ii) in the case where the following conditions are satisfied:

(3.13.4) X is the open polydisc D^n;

(3.13.5) $Y = \bigcup_{1\leqslant i\leqslant k} Y_i$, with $Y_i = \mathrm{pr}_i^{-1}(0)$; and

(3.13.6) $\mathcal{V}$ is the constant vector bundle defined by a vector space V, and the connection is of the form

$$\Gamma = \sum_i \Gamma_i \frac{\mathrm{d}z_i}{z_i}$$

with $\Gamma_i \in \mathrm{End}(V)$ and $\Gamma_i = 0$ for $i > k$.

Since the connection is integrable, the Γ_i pairwise commute, and there exists a finite filtration G of V that is stable under the Γ_i, and such that $\dim \mathrm{Gr}_G^l(V) \leqslant 1$. By the first reduction, we can assume that $V = \mathbb{C}$, in which case the Γ_i can be identified with a scalar γ_i. The bundle $\mathcal{V}$ with connection is then the external tensor product of the bundles $(\mathcal{O}, \gamma_i \mathrm{d}z/z)$ on D. The second reduction allows us to assume that $n = 1$. If $k = 0$, i.e. if $Y = \varnothing$, then $\Omega_X^\bullet\langle Y\rangle = j_*^{\mathrm{m}}\Omega_{X^*}^\bullet$ and $F = P$. If $k = 1$, i.e. if $Y = \{0\}$, then

a) $P^i(j_*^{\mathrm{m}}\Omega_{X^*}^\bullet(\mathcal{V})) = 0$ for $i > -1$; p. 83

b) $P^{-1}(j_*^{\mathrm{m}}\Omega_{X^*}^p(\mathcal{V}))$ is equal to 0 if $p = 0$, and to $\Omega_X^1\langle Y\rangle(\mathcal{V})$ if $p = 1$;

c) $\mathrm{Gr}_P^0(j_*^{\mathrm{m}}\Omega_{X^*}^\bullet(\mathcal{V}))$ is the complex

$$\frac{1}{z}\mathcal{O} \xrightarrow{\partial_z + \gamma} \frac{1}{z^2}\mathcal{O} \Big/ \frac{1}{z}\mathcal{O}$$

and if $\gamma - 1 \neq 0$ then $\mathrm{Coker}(\mathrm{d}) = 0$ and $\mathrm{Ker}(\mathrm{d}) = \mathcal{V} = \Omega_X^0\langle Y\rangle(\mathcal{V})$; and

d) $\mathrm{Gr}_P^{-n}(j_*^{\mathrm{m}}\Omega_{X^*}^\bullet(\mathcal{V}))$, for $n > 0$, is the complex

$$\frac{1}{z^{n+1}}\mathcal{O} \Big/ \frac{1}{z^n}\mathcal{O} \xrightarrow{\partial_z + \gamma} \frac{1}{z^{n+2}}\mathcal{O} \Big/ \frac{1}{z^{n+1}}\mathcal{O}.$$

This satisfies (3.13.ii) case by case.

General case. Since the problem is local, we can suppose that conditions (3.13.4) and (3.13.5) are satisfied, and it suffices to prove that the germ at 0 of (3.13.1) is a quasi-isomorphism.

For $0 < |t| \leqslant 1$, let $\mathcal{V}_t$ be the bundle with connection given by the inverse image of $\mathcal{V}$ under the homothety H_t with ratio t. As $t \to 0$, the $\mathcal{V}_t$ "tend" to the constant vector bundle $\mathcal{V}_0$ defined by the fibre V_0 of $\mathcal{V}$ at 0, endowed with a connection satisfying (3.13.4), (3.13.5), and (3.13.6).

More precisely, let H and i_t be the morphisms

$$\begin{aligned} H\colon D^n \times D \to D^n\colon &\quad (x,t) \mapsto t \cdot x \\ i_t\colon D^n \to D^n \times D\colon &\quad x \mapsto (x,t). \end{aligned}$$

Then $H_t = H \circ i_t$. The inverse image of the connection ∇ on $\mathcal{V}$ is ∇_1 on $H^*\mathcal{V}|H^{-1}(X^*)$. The corresponding relative connection (relative to pr_2) extends to $H^*(\mathcal{V})|X^* \times D$. If we set $\mathcal{V}_t = i_t^* H^*\mathcal{V}$, then, for $t \neq 0$, we have an isomorphism of bundles with connections

$$\mathcal{V}_t \cong H_t^*(\mathcal{V}). \tag{3.13.7}$$

For $t = 0$, we have an isomorphism of vector bundles p. 84

$$\mathcal{V}_0 = H_0^*(\mathcal{V}) = \mathcal{O}_X \otimes_{\mathbb{C}} V_0 \tag{3.13.8}$$

and the connection on $\mathcal{V}_0|X^*$ satisfies (3.13.4), (3.13.5), and (3.13.6).

The relative version of (3.12) gives a morphism of filtered complexes

$$\varphi\colon \big(\Omega_{X\times D/D}^\bullet\langle Y \times D\rangle(H^*\mathcal{V}), F\big) \to \big(j_*^{\mathrm{m}}\Omega_{X^*\times D}^\bullet(H^*\mathcal{V}), P\big). \tag{3.13.9}$$

The associated graded complexes are flat over D (via pr_2), and their homogeneous graded components are coherent, and the differentials are $\mathcal{O}_{X\times D}$-linear. We already know that $i_0^* \mathrm{Gr}(\varphi)$ is a quasi-isomorphism. It thus follows that $i_t^* \mathrm{Gr}^p(\varphi)$ (the arrow (3.13.1) for $\mathcal{V}_t$) is a quasi-isomorphism near to 0, for t small enough. Since the $\mathcal{V}_t$ are isomorphic to one another, close to 0, for $t \neq 0$ (3.13.7), $i_1^* \mathrm{Gr}^p(\varphi)$ is a quasi-isomorphism near to 0, which proves .

□

Corollary 3.14. *Let X be a smooth scheme over $\mathbb{C}$, Y a normal crossing divisor in X, j the inclusion of $X^* = X \setminus Y$ into X, $\mathcal{V}$ a vector bundle on X, and Γ an integrable connection on $\mathcal{V}|X^*$ that presents a logarithmic pole along Y. Suppose that the residues of the connection along Y do not admit any strictly positive integer as an eigenvalue. Then*

(i) the homomorphism of complexes

$$i : \Omega_X^\bullet\langle Y\rangle(\mathcal{V}) \to j_*\Omega_{X^*}^\bullet(\mathcal{V})$$

induces an isomorphism on the cohomology sheaves (for the Zariski topology); and

(ii) more precisely, i is injective, and there exists an exhaustive increasing filtration of the complex Coker(i) *whose successive quotients are acyclic complexes whose differentials are linear.*

Proof. The filtration P of (3.12) has an evident algebraic analogue, which also satisfies conditions a) to d) of (3.12). The corollary follows from the statement, which is more precise than (ii), saying that the complexes

$$G^i = \mathrm{Gr}_P^i \left(j_*\Omega_{X^*}^\bullet(\mathcal{V}) \big/ \Omega_X^\bullet\langle Y\rangle(\mathcal{V}) \right)$$

are acyclic. These complexes have $\mathcal{O}_X$-linear differentials, and, by (3.13), the $(G^i)^{\mathrm{an}}$ are acyclic. By the flatness of $\mathcal{O}_{X^{\mathrm{an}}}$ over $\mathcal{O}_X$, the G^i are thus acyclic, which finishes the proof. *p. 85*

□

Corollary 3.15. *Under the hypotheses of (3.14), we have*

$$\mathbb{H}^k\left(X, \Omega_X^\bullet\langle Y\rangle(\mathcal{V})\right) \xrightarrow{\sim} \mathbb{H}^k\left(X^*, \Omega_{X^*}^\bullet(\mathcal{V})\right).$$

Proof. Since the morphism j is affine, we have

$$\mathbf{R}^k j_*\Omega_{X^*}^p(\mathcal{V}) = 0 \quad \text{for } k > 0$$

and so

$$\mathbb{H}^\bullet\left(X, j_*\Omega_{X^*}^\bullet(\mathcal{V})\right) \xrightarrow{\sim} \mathbb{H}^\bullet\left(X^*, \Omega_{X^*}^\bullet(\mathcal{V})\right).$$

Also, by (3.14.i), we have

$$\mathbb{H}^\bullet\left(X, \Omega_X^\bullet\langle Y\rangle(\mathcal{V})\right) \xrightarrow{\sim} \mathbb{H}^\bullet\left(X, j_*\Omega_{X^*}^\bullet(\mathcal{V})\right)$$

whence the corollary. □

Remark 3.16. It is easy to generalise (3.13) and (3.14) to the relative situation, where $f : X \to S$ is a smooth morphism (with S an analytic space, or scheme of characteristic 0), and Y is a relative normal crossing divisor.

II.4 Regularity in dimension n

[Translator] The proof of Theorem 4.1 has been replaced with the proof given in the errata, which also cites the following:

> N. Katz. The regularity theorem in algebraic geometry. *Actes du Congrès intern. math.* **1** (1970), 437–443.

Note that pages 87 and 88 are thus missing from this translation.

Theorem 4.1. *Let X be a complex-analytic space, Y a closed analytic subset of X such that $X^* = X \setminus Y$ is smooth, X' the normalisation of X, and Y' the inverse image of Y in X'. Let $\mathcal{V}$ be a vector bundle on X^* that is meromorphic along Y, and let ∇ be a connection on $\mathcal{V}$. Then the following conditions are equivalent:*

(i) there exists an open subset U of Y' that contains a point of each codimension 1 component of Y', and an isomorphism φ from a neighbourhood of U in X' to $U \times D$ (where D is the unit disc) that induces the identity map from U to $U \times \{0\}$, such that, for all $u \in U$, the restriction of $\varphi^*\mathcal{V}$ to $\{u\} \times D$ is regular at 0; p. 86

(ii) for every map $\varphi\colon D \to X$ with $\varphi^{-1}(Y) = \{0\}$, the inverse image of V under φ is regular; and

(iii) the (multiform) horizontal sections of $\mathcal{V}$ are of moderate growth along Y.

By (i), these conditions hold "in codimension 1 at infinity" on the normalisation of X. If Y is a smooth normal crossing divisor in X, then the above three conditions are also equivalent to the following:

(iv) for all $y \in Y$, there exists an open neighbourhood U of y, and a basis $e\colon \mathcal{O}^d \to \mathcal{V}$ of $\mathcal{V}$ on $U \setminus Y$ that is meromorphic along Y, such that the connection matrix (which is a matrix of differential forms) presents at worst a logarithmic pole along Y.

Proof. We will use constructions (5.1) to (5.5); the reader can verify the lack of circularity.

The theorem is local along Y, which allows us to suppose that

a) the meromorphic structure of $\mathcal{V}$ along Y is defined by a coherent extension $\widetilde{\mathcal{V}}$ of the vector bundle $\mathcal{V}$ on X; and

b) there exists a resolution of singularities $\pi\colon X_1 \to X$ such that X_1 is smooth, π is proper, $\pi^{-1}(X^*) \xrightarrow{\sim} X^*$, and such that $Y_1 = X_1 \setminus \pi^{-1}(X^*)$ is a normal crossing divisor in X_1.

We can think of $\mathcal{V}$ as a vector bundle with integrable connection on $X_1^* = \pi^{-1}(X^*)$, and $\widetilde{\mathcal{V}}$ as its canonical extension on X_1. The vector bundle $\mathcal{V}_0 = \pi_*\widetilde{\mathcal{V}}$ is a coherent extension of $\mathcal{V}$ on X. We will show that conditions (i), (ii), and (iii) are all equivalent to the following:

(v) The extensions $\widetilde{\mathcal{V}}$ and $\mathcal{V}_0$ of $\mathcal{V}$ are meromorphically equivalent.

First we prove that (i) $\Longrightarrow$ (v). If X is of dimension 1, then this is a consequence of (1.20). Let u be the identity map from $\mathcal{V}_0|X^*$ to $\widetilde{\mathcal{V}}|X^*$. We need to show that, for every open subset $W \subset X$, for every linear form $w \in \mathrm{H}^0(W, \mathcal{V}^\vee)$, and for every local section $e \in \mathrm{H}^0(W, \mathcal{V}_0)$, the function $f = \langle w, u(e) \rangle$ is meromorphic along $Y \cap W$.

Suppose that X is smooth, that Y is a smooth divisor in X, and that $X_1 = X$. The fact that f is meromorphic follows from the already discussed dimension 1 case, and from two applications of the following lemma (once to show that f is meromorphic along U, and once again to show that f is meromorphic along Y).

Lemma 4.1.1. *Let f be an analytic function on $D^{m+1} \setminus (\{0\} \times D^m)$. Suppose that there exists a non-empty open subset U of D^m such that, for all $u \in U$, $f|(D^* \times \{u\})$ is meromorphic at 0. Then there exists some n such that f has a pole of order at most n along $\{0\} \times D^m$.*

Proof. Let $F_n \subset D^m$ be the set of u such that $f|D^* \times \{u\}$ presents at worst a pole of order n at 0. Let

$$A_k(u) = \oint f(z,u) z^k \, \mathrm{d}z.$$

Then $A_k(u)$ is holomorphic, and F_n is defined by the equations $A_k(u) = 0$ for $k \geqslant n$. By hypothesis, the union of the closed subsets F_n has an interior point. By Baire, there exists some n such that F_n has an interior point, and $A_k(u)$ is thus zero on an open subset (and thus everywhere) for all $k \geqslant n$. But then f has at worst a pole of order n along $\{0\} \times D^m$. □

To pass from here to the case where X is normal, we note that the above conditions are then satisfied outside of a subset Z of Y of codimension $\geqslant 2$ in X. We conclude by noting that a function f on $X \setminus Y$ which is meromorphic along Y outside of Z is meromorphic along Y. Indeed, the proof of (4.1.1) shows that, locally on Y, the product of f with a high enough power g^k of a function that vanishes on Y is holomorphic on $X \setminus Z$, and this product extends to a holomorphic function on X.

In the general case, we note that condition (i) (resp. (v)) is equivalent to condition (i) (resp. (v)) on the normalisation of X.

It is trivial that (ii) $\Longrightarrow$ (i), and it follows from (1.19) that (iii) $\Longrightarrow$ (i).

Under the hypotheses of (iv), and for $X_1 = X$, it is clear that (v) $\Longrightarrow$ (iv); by (5.5.i), (v) $\Longrightarrow$ (iii); since the inverse image of a differential form that presents at worst a simple pole also presents at worst a simple pole, (iv) $\Longrightarrow$ (ii). Under these hypotheses, claims (i) to (v) are thus equivalent.

It follows from (2.19) and from the above that (v) $\Longrightarrow$ (iii). To prove that (v) $\Longrightarrow$ (ii), we can either use (1.19) and (2.19) to show that (iii) $\Longrightarrow$ (ii), or we can note that condition (ii) for $\mathcal{V}$ on $X^* \subset X$ is equivalent to condition (ii) for $\pi^{-1}\mathcal{V}$ on $X^* \subset X_1$. Conditions (i), (ii), (iii), and (v) are thus equivalent. In particular, condition (v) is independent of the choice of X_1; we thus deduce that, under the hypotheses of (iv), (iv) $\Longleftrightarrow$ (v), and this finishes the proof. □

We note that the above proof already contains the essential part of the proof of (5.7) and (5.9) (the existence theorem). *p. 89*

Definition 4.2. Under the hypotheses of (4.1), we say that $(\mathcal{V}, \nabla)$ is *regular along Y* if any of the equivalent conditions of (4.1) are satisfied.

Proposition 4.3. *With the hypotheses and notation of (2.19), let $\mathcal{V}$ be a vector bundle on X_2^* that is meromorphic along Y_2, endowed with an integrable connection. Then*

(a) if $\mathcal{V}$ is regular, then $f^*\mathcal{V}$ is regular; and

(b) if condition (2.19.a) is satisfied, and if $f^*\mathcal{V}$ is regular, then $\mathcal{V}$ is regular.

Proof. By (2.19), this is clear from (4.1.iii). □

Proposition 4.4. *Let $\mathcal{V}$ be a vector bundle on a smooth separated complex-algebraic variety X. Let $\overline{X}$ be a compactification of X, so that $\mathcal{V}^{\mathrm{an}}$ is meromorphic along $Y = \overline{X} \setminus X$. Let ∇ be a connection on $\mathcal{V}^{\mathrm{an}}$. Then the following conditions are equivalent:*

(i) $\mathcal{V}^{\mathrm{an}}$ is regular along Y; and

(ii) for every smooth algebraic curve C on X (and locally closed in X), $\mathcal{V}|C$ is regular (1.21).

If $\overline{X}$ is normal, then the above two conditions are also equivalent to the following:

(iii) ∇ is algebraic, and, for every generic point η of a codimension 1 component of Y, there exists an algebraic vector field v on a neighbourhood of η, with v transversal to Y (so that the triple $(\mathcal{O}_\eta, \mathcal{O}_\eta, \partial_v)$ satisfies (1.4.1)), such that $\mathcal{V}$ induces, over the field of fractions K of $\mathcal{O}_\eta$, endowed with ∂_v, a vector space with regular connection, in the sense of (1.11); and p. 90

(iii') idem. *for every field v of this type.*

Proof. We have (ii) $\implies$ (4.1.i) $\implies$ (4.1.ii) $\implies$ (ii). Also, (4.1.iv) implies that ∇ is meromorphic in codimension 1 on $\overline{X}$, and thus meromorphic, and thus algebraic (by GAGA). We thus have

$$\text{(iii')} \implies \text{(iii)} \implies \text{(4.1.i)} \implies \text{(4.1.iv)} \implies \text{(iii')}.$$

□

Definition 4.5. Under the hypotheses of (4.4), we say that $(\mathcal{V}, \nabla)$ is *regular* if any of the equivalent conditions of (4.4) are satisfied.

If $(\mathcal{V}, \nabla)$ is a vector bundle with integrable algebraic connection on X, then it is clear, by (4.4.ii), that the regularity of ∇ is a purely algebraic condition, independent of the choice of compactification. We can, in many different ways, define regularity when X is a smooth scheme of finite type over a field k of characteristic 0. For example, we can take (4.4.ii) or (4.4.iii) as a definition. We will restrict ourselves in what follows to the case where $k = \mathbb{C}$. By the Lefschetz principle, this does not reduce the level of generality.

Proposition 4.6. *Let X be a (smooth) complex-algebraic variety.*

(i) If $\mathcal{V}' \to \mathcal{V} \to \mathcal{V}''$ is a horizontal exact sequence of vector bundles with integrable connections on X, and if $\mathcal{V}'$ and $\mathcal{V}''$ are regular, then $\mathcal{V}$ is regular.

(ii) If $\mathcal{V}_1$ and $\mathcal{V}_2$ are vector bundles with regular integrable connections on X, then $\mathcal{V}_1 \otimes \mathcal{V}_2$, $\underline{\mathrm{Hom}}(\mathcal{V}_1, \mathcal{V}_2)$, $\mathcal{V}_1^\vee$, and $\bigwedge^p \mathcal{V}_1$ are all regular.

(iii) Let $f\colon X \to Y$ be a morphism of smooth schemes over $\mathbb{C}$, and $\mathcal{V}$ a vector bundle with integrable connection on Y. If $\mathcal{V}$ is regular, then $f^*\mathcal{V}$ is regular. Conversely, if $f^*\mathcal{V}$ is regular and f is dominant, then $\mathcal{V}$ is regular.

Proof. Claims (i) and (ii) follow immediately from the definition, by (4.4.ii) and (1.13). It is *p. 91*
clear, by (4.4.iii), that regularity, being satisfied in codimension 1 at infinity, is a *birational notion*. This allows us to replace "f is dominant" in (iii) by "f is surjective". We then apply (4.4.ii) and (1.13.iii) by noting that, for f surjective, for every curve C on Y, there exists a curve C' on X such that $\overline{f(C')} \supset C$. □

II.5 Existence theorem

5.1. Let D be the open unit disk, $D^* = D \setminus \{0\}$, $X = D^{n+m}$, $Y_i = \mathrm{pr}_i^{-1}(\{0\})$, and $Y = \bigcup_{i=1}^n Y_i$. Set $X^* = X \setminus Y = (D^*)^n \times D^m$. We have

$$\pi_1(X^*) = \pi_1(D^*)^n = \mathbb{Z}^n \tag{5.1.1}$$

via the identification $\pi_1(D^*) \simeq \mathbb{Z}$ from (1.15). We denote by T_i the element of the abelian group $\pi_1(X^*)$ that corresponds, by (4.1.1), to the i-th basis vector of $\mathbb{Z}^n$.

A local system $\mathcal{V}$ on X^* is said to be *unipotent along* Y if the fundamental group $\pi_1(X^*)$ acts on this local system (I.1.5) by unipotent transformations. We will use the same terminology for when $\mathcal{V}$ is a vector bundle endowed with an integrable connection on X^* (via the dictionary in (I.2.17)). Since $\pi_1(X^*)$ is abelian, generated by the "monodromy transforms" T_i, it is equivalent to ask for the T_i to act unipotently.

In the following proposition, we denote by $\|\ \|$ an arbitrary norm on X^* with respect to Y, for example

$$\|z\| = \frac{1}{d(z,Y)} \quad \text{or} \quad \|z\| = \frac{1}{\prod_{i=1}^n |z_i|}.$$

Proposition 5.2. *With the notation of (4.1), let $\mathcal{V}$ be a vector bundle with integrable connection on X^* that is unipotent along Y. Then*

(a) There exists a unique extension $\widetilde{\mathcal{V}}$ of the vector bundle $\mathcal{V}$ to a vector bundle on X that satisfies the following conditions: *p. 92*

(i) Every (multiform) horizontal section of $\mathcal{V}$ has, as a multiform section of $\widetilde{\mathcal{V}}$ on X^*, a growth of at most $\mathcal{O}((\log\|x\|)^k)$ (for large enough k) near every compact subset of Y.

(ii) Similarly, every (multiform) horizontal section of $\mathcal{V}^\vee$ has a growth of at most $\mathcal{O}((\log\|x\|)^k)$ (for large enough k) near every compact subset of Y.

(b) The combination of conditions (i) and (ii) above is equivalent to the following conditions:

(i) The matrix of the connection of $\mathcal{V}$, in an arbitrary local basis of $\widetilde{\mathcal{V}}$, has at worst a logarithmic pole along Y.

(ii) The residue $\mathrm{Res}_i(\Gamma)$ of the connection along Y_i (for $1 \leqslant i \leqslant n$) is nilpotent.

(c) Let e be a (multiform) horizontal basis of $\mathcal{V}$. Then the sections of $\widetilde{\mathcal{V}}$ over X can be identified with the sections of $\mathcal{V}$ over X^* whose coordinates in the basis e are (multiform) functions with growth at most $\mathcal{O}((\log\|x\|)^k)$ (for large enough k) near every compact subset of Y.

(d) Every horizontal morphism $f\colon \mathcal{V}_1 \to \mathcal{V}_2$ can be extended to $\tilde{f}\colon \widetilde{\mathcal{V}}_1 \to \widetilde{\mathcal{V}}_2$. The functor $\mathcal{V} \mapsto \widetilde{\mathcal{V}}$ is exact, and compatible with $\otimes$, $\underline{\mathrm{Hom}}$, $\bigwedge^p$, ...

We call $\widetilde{\mathcal{V}}$ the *canonical extension* of $\mathcal{V}$.

Proof. —

(a) Let $e\colon \mathcal{O}^n \to \mathcal{V}$ be a (multiform) horizontal basis of $\mathcal{V}$, and $\mathcal{V}_1$ an extension of $\mathcal{V}$. Condition (i) implies that $e\colon \mathcal{O}^n \to \mathcal{V}_1|X^*$ has growth $\mathcal{O}((\log\|x\|)^k)$. Condition (ii) implies that the dual basis $e'\colon \mathcal{O}^n \to \mathcal{V}_1|X^*$ has growth at most $\mathcal{O}((\log\|x\|)^k)$. If $\mathcal{V}_1$ and $\mathcal{V}_2$ are two extensions of $\mathcal{V}$ satisfying (i) and (ii), then the identity map i of $\mathcal{V}$ fits into the commutative diagram

$$\begin{array}{ccc} \mathcal{V}_1|X^* & \xrightarrow{\ i\ } & \mathcal{V}_2|X^* \\ {\scriptstyle e^{-1}}\big\downarrow & & \big\downarrow{\scriptstyle e} \\ \mathcal{O}^n & = & \mathcal{O}^n \end{array}$$

By hypothesis, e^{-1} and e, and thus i, have growth $\mathcal{O}((\log\|x\|)^k)$, and so i is regular, and so i^{-1} is also regular: the identity of $\mathcal{V}$ extends to an isomorphism between $\mathcal{V}_1$ and $\mathcal{V}_2$. This proves the uniqueness of $\widetilde{\mathcal{V}}$. *p. 93*

(b) Let $\mathcal{V}_0$ be a vector space endowed with a unipotent action of $\pi_1(X^*)$, and let $-2\pi i U_i$ be the nilpotent determination of the logarithm of the action of T_i, i.e.

$$U_i = \frac{1}{2\pi i}\sum\frac{(I-T_i)^k}{k}. \tag{5.2.1}$$

Let $\widetilde{\mathcal{V}}_0$ be the vector bundle on X defined by $\mathcal{V}_0$, and endowed with the connection matrix

$$\Gamma = \sum U_i\frac{\mathrm{d}z_i}{z_i}. \tag{5.2.2}$$

The (multiform) horizontal sections of $\widetilde{\mathcal{V}}_0$ are of the form

$$v(z) = \exp\left(-\sum(\log z_i)U_i\right)(v_0). \tag{5.2.3}$$

The exponential series reduces here to a finite sum, and thus to a polynomial in the $\log z_i$.

The vector bundle $\widetilde{\mathcal{V}}_0$ satisfies (i), (ii), (iii), and (iv). Further, if e_0 is a basis of $\mathcal{V}_0$, then the relation between the coordinates of a section v of $\widetilde{\mathcal{V}}_0$ in the basis e_0, or in the horizontal basis e_1 given by (5.2.3), is given by

$$\begin{cases} e_0 & = \exp\left(\sum_{i=1}^n (\log z_i) U_i\right) e_1 \\ e_1 & = \exp\left(\sum_{i=1}^n (\log z_i) U_i\right) e_0 \end{cases}$$

and (c) is satisfied. Finally, $\widetilde{\mathcal{V}}_0$ is an exact functor in $\mathcal{V}_0$, and its construction is compatible with $\otimes$, $\underline{\mathrm{Hom}}$, $\wedge^p$, ... Given the dictionary in (I.1.2), this proves (a), (c), and (d), and shows that (i)+(ii) $\Longrightarrow$ (iii)+(iv).

(c) Let $\mathcal{V}_1$ be an extension of $\mathcal{V}$ satisfying (iii) and (iv). To prove (i), we reduce to the case where $\mathcal{V}_1 \sim \mathcal{O}^n$ is free. Let Γ be the connection matrix, and write

$$\Gamma = \sum \Gamma_{0,i} \frac{\mathrm{d}z_i}{z_i} + \Gamma'' = \Gamma' + \Gamma''$$

with $\Gamma_{0,i}$ constant, and Γ'' holomorphic. Let e be a (multiform) horizontal basis for the connection Γ' (which is of the type considered in (b)). If $a \cdot e$ is a (multiform) horizontal basis for Γ, then

$$\nabla(a \cdot e) = \mathrm{d}a \cdot \nabla e + a \cdot \nabla e = 0$$

whence the bound

$$|\mathrm{d}a| \leqslant C^{te} |a| (\log \|z\|)^k.$$

This proves that $\mathcal{V}_1$ satisfies (i), and (ii) follows by considering $\mathcal{V}_1^\vee$.

□

5.3. If we do not suppose $\mathcal{V}$ to be unipotent along Y, then it is still possible to "make" an extension of $\mathcal{V}$ on X, but its construction, which is much more arbitrary, depends on the choice of a section τ of the projection of $\mathbb{C}$ to $\mathbb{C}/\mathbb{Z}$. Choosing τ is equivalent to choosing a logarithm function: we set

$$\log_\tau(x) = 2\pi i \tau \left(\frac{1}{2\pi i} \log z \right).$$

One of the least bad choices is

$$0 \leqslant \Re(\tau) < 1. \tag{5.3.1}$$

Proposition 5.4. (Manin [19]). *Let τ be as in (5.3), and let $\mathcal{V}$ be a vector bundle with an integrable connection on X^*. Then there exists a unique extension $\widetilde{\mathcal{V}}(\tau)$ of the vector bundle $\mathcal{V}$ to a vector bundle on X that satisfies the following conditions:*

(i) The connection matrix of $\mathcal{V}$, in an arbitrary local basis of $\widetilde{\mathcal{V}}(\tau)$, presents at worst a logarithmic pole along Y;

(ii) The residue $\mathrm{Res}_i(\Gamma)$ of the connection along Y_i (for $1 \leqslant i \leqslant n$) has eigenvalues in the image of τ.

The extension $\widetilde{\mathcal{V}}(\tau)$ of $\mathcal{V}$ is functorial and exact in $\mathcal{V}$.
(We note that the construction of $\widetilde{\mathcal{V}}(\tau)$ is not, in general, compatible with $\otimes$.) *p. 95*

Proof. For every homomorphism $\lambda\colon \pi_1(X^*) \to \mathbb{C}^*$, let $U_{\lambda,\tau}$ (or simply U_λ) be the vector bundle $\mathscr{O}$ on X endowed with the connection matrix

$$\Gamma_\lambda = \sum \frac{-1}{2\pi i}(\log_\tau \lambda(T_i))\frac{\mathrm{d}z_i}{z_i}.$$

By construction, U_λ satisfies (i) and (ii), and admits λ as its monodromy.

Let $\mathcal{V}$ be an arbitrary vector bundle with integrable connection on X^*. Then $\mathcal{V}$ admits a unique decomposition

$$V \simeq \bigoplus_\lambda (U_\lambda | X^*) \otimes \mathcal{V}_\lambda$$

with $\mathcal{V}_\lambda$ unipotent along Y. The direct factor $(U_\lambda|X^*)\otimes\mathcal{V}_\lambda$ of $\mathcal{V}$ is the largest sub-bundle of $\mathcal{V}$ on which the $T_i - \lambda(T_i)$ are nilpotent. The extension (cf. (5.2))

$$\widetilde{\mathcal{V}}_\tau = \bigoplus_\lambda U_\lambda \otimes \widetilde{\mathcal{V}}_\lambda \tag{5.4.1}$$

of $\mathcal{V}$ satisfies (i) and (ii), and is exact and functorial in $\mathcal{V}$.

To prove that the extension problem has only one solution, it suffices to prove this locally on Y outside of a subset of codimension $\geqslant 2$ in X. With the notation of (5.1), this allows us to reduce to the case where $n = 1$. So let $\mathcal{V}'$ be an extension of $\mathcal{V}$ satisfying (i) and (ii). By (3.11), the monodromy transformation T extends to an automorphism of $\mathcal{V}'$. The characteristic polynomial of T is constant; the vector bundle $\mathcal{V}'$ thus decomposes uniquely into its T-stable vector sub-bundles on which an endomorphism $(T-\lambda)$ is nilpotent (i.e. the generalised eigenspaces of T). We can thus write $\mathcal{V}'$ uniquely as

$$\mathcal{V}' = \bigoplus_\lambda U_\lambda \otimes \mathcal{V}'_\lambda$$

with $\mathcal{V}'_\lambda|X^*$ unipotent along Y. By construction, $\mathcal{V}'_\lambda$ satisfies (iii) and (iv) of (5.2), and so, by (5.2), $\mathcal{V}'_\lambda = \widetilde{\mathcal{V}}_\lambda$. We thus have that $\mathcal{V}' = \widetilde{\mathcal{V}}(\tau)$ (5.4.1). □

Remarks 5.5. —

(i) If we define τ by (5.3.1), then $\widetilde{\mathcal{V}}(\tau)$ satisfies (5.2) (c). We also call this extension the *canonical extension* of $\mathcal{V}$. *p. 96*

(ii) An extension of the form $\widetilde{\mathcal{V}}(\tau)$ has the property that, in a suitable basis of $\widetilde{\mathcal{V}}(\tau)$, the connection matrix takes the form

$$\Gamma = \sum_i \Gamma_i \frac{\mathrm{d}z_i}{z_i}$$

where the Γ_i are connection matrices that commute pairwise.

Corollary 5.6. (N. Katz). *Let $\mathcal{V}$ be a vector bundle on the unit disc D, and Γ a connection on $\mathcal{V}|D^*$ whose matrix presents a simple pole at* 0*. Suppose that, for any distinct eigenvalues α and β of* $\mathrm{Res}(\Gamma)$*, we have $\alpha - \beta \in \mathbb{Z}$. Then the monodromy transformation T is conjugate, in the linear group, to* $\exp(-2\pi i\,\mathrm{Res}(\Gamma))$.

Proof. By (5.4), $\mathcal{V}$ is of the form $\widetilde{\mathcal{V}_0}(\tau)$ for some suitable $\mathcal{V}_0$; we conclude by direct calculation (1.17.1). □

Proposition 5.7. *Let X be a complex-analytic space,*

!TO-DO!

II.6 Comparison theorem

!TO-DO!

II.7 Regularity theorem

!TO-DO!

Chapter III

Applications and examples

III.1 Functions in the Nilsson class

1.1. Let X be a non-singular complex algebraic variety that is connected and endowed with a base point x_0. We denote by $\widetilde{X}$ the universal cover of (X, x_0), and by $\widetilde{x}_0$ the base point of $\widetilde{X}$. Suppose that we have a given complex representation W_0 of finite dimension d of $\pi_1(X, x_0)$ endowed with a cyclic vector w_0. We denote by W the corresponding local system (I.1.4), and by $\mathscr{W}$ the algebraic vector bundle with regular integrable connection endowed with an isomorphism of $\pi_1(X, x_0)$-representations (II.5.7) p. 122

$$\mathscr{W}_{x_0} \simeq W.$$

Finally, we denote by w the multiform horizontal section of $\mathscr{W}^{\mathrm{an}}$ with base determination w_0.

Definition 1.2. A section of an *algebraic* vector bundle $\mathscr{V}$ on X is said to be *in the Nilsson class* if it is a multiform holomorphic section of finite determination that is of moderate growth at infinity (II.2.23.iv).

If $\mathscr{V} = \mathscr{O}$, then we speak of *functions in the Nilsson class*.

The two following theorems will be proven simultaneously in (1.5). The first says that, for a function *of finite determination*, various variants of the "moderate growth at infinity" condition are equivalent.

Theorem 1.3. *Let s be a multiform holomorphic section of finite determination of an algebraic vector bundle on X. Then the following conditions are equivalent:*

(i) s is in the Nilsson class; and

(ii) the restriction of s to every (locally closed) smooth algebraic curve along X is in the Nilsson class.

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If X is a Zariski open subset of a compact normal variety $\overline{X}$, then the two conditions above are also equivalent to the following:

(iii) *every irreducible component of $\overline{X} \setminus X$ of codimension 1 in $\overline{X}$ contains a non-empty open subset U along which s is of moderate growth.*

In the above, we do not lose any generality in supposing, in (iii), that $X \cup U \subset \overline{X}$ is smooth, and that U is a smooth divisor there. Unlike (i), conditions (ii) and (iii) do not make any reference to the theory of Lojasiewicz. It follows from (iii) that, if $\operatorname{codim}(\overline{X} \setminus X) \geqslant 2$, then a function of finite determination is automatically in the Nilsson class.

For X of dimension 1, the following theorem is due to Plemelj [23].

Theorem 1.4. *Let $\mathcal{V}$ be an algebraic vector bundle on X. The "evaluation at w" function, that sends each (algebraic) $f \in \operatorname{Hom}(\mathcal{W},\mathcal{V})$ to the section $f(w)$ of $\mathcal{V}^{\mathrm{an}}$ gives a bijection between $\operatorname{Hom}(\mathcal{W},\mathcal{V})$ and the set of sections of $\mathcal{V}$ in the Nilsson class that are of monodromy subordinate to* (W_0, w_0).

1.5. Here we prove (1.3) and (1.4). We have already seen (in (I.6.11)) that the function $E_w\colon f \mapsto f(w)$ identifies $\operatorname{Hom}(\mathcal{W}^{\mathrm{an}},\mathcal{V}^{\mathrm{an}})$ with the set of multiform holomorphic sections of $\mathcal{V}^{\mathrm{an}}$ of finite determination with monodromy subordinate to (W_0,w_0). It thus remains to prove that f is algebraic if and only if $f(w)$ satisfies (1.3.i) (resp. (1.3.ii), resp. (1.3.iii)).

By (II.4.1.iii), the "section" w of $\mathcal{W}$ is in the Nilsson class, so that, if f is algebraic, then $f(w)$ satisfies (1.3.i). Trivially, (1.3.i) $\Longrightarrow$ (1.3.ii) and (1.3.i) $\Longrightarrow$ (1.3.iii).

Let $e\colon \mathcal{O}^d \to \mathcal{W}$ be a multiform basis of $\mathcal{W}$ consisting of determinations of w. Since $\mathcal{W}$ is regular, e^{-1} is of moderate growth at infinity. For $f\colon \mathcal{W}^{\mathrm{an}} \to \mathcal{V}^{\mathrm{an}}$, the $f(e_i)$ are determinations of $f(w)$. From this, and from the equation $f = fee^{-1}$, we deduce that p. 124

a) if $f(w)$ satisfies (1.3.ii), then the restriction of $f \in \mathrm{H}^0(\underline{\operatorname{Hom}}(\mathcal{W},\mathcal{V})^{\mathrm{an}})$ to any curve is of moderate growth; and

b) if $f(w)$ satisfies (1.3.iii), then f is of moderate growth near a non-empty open subset of each irreducible component of $\overline{X} \setminus X$ of codimension 1.

By (II.4.1.1), under each of these hypotheses, f is algebraic.

Corollary 1.5. *Under the hypotheses of (1.4), if X is affine, with coordinate ring A, and if $\mathcal{V}$ is of rank m, then the set of sections of $\mathcal{V}$ in the Nilsson class that are of monodromy subordinate to* (W_0, w_0) *is a* projective *A-module of rank dm.*

Remark 1.6. A *meromorphic function in the Nilsson class* is, by definition, a section of some sheaf $\mathcal{O}(D)$, for D a sufficiently positive divisor, in the Nilsson class (where $\mathcal{O}(D)$ is the sheaf of meromorphic functions f such that $\operatorname{div}(f) \geqslant -D$). It follows from (1.5) that the set of meromorphic functions in the Nilsson class that are of monodromy subordinate to (W_0,w_0) is a vector space of dimension d over the field of rational functions on X.

1.7. Let $f\colon X \to S$ be a smooth morphism, with S smooth. By (1.4), the set of relative differential p-forms on X that are in the Nilsson class and of monodromy subordinate to (W_0,w_0) can be identified with the space

$$\mathrm{H}^0\left(S, \operatorname{Ker}\left(\mathrm{d}\colon f_*\Omega^p_{X/S}(\mathcal{W}^\vee) \to f_*\Omega^{p+1}_{X/S}(\mathcal{W}^\vee)\right)\right).$$

Let U' be a dense Zariski open subset of S such that, over U', f is locally C^∞-trivial. The *homology* groups $\mathrm{H}_p(X_S^{\mathrm{an}}, W)$ then form a local system $\mathscr{H}$ on U'.

We denote by $\langle -,- \rangle$ the pairing of sheaves on a small enough dense Zariski open subset $U \subset U'$ (II.6.13)

$$\begin{aligned} \mathscr{H} \otimes \mathrm{Ker}\big(\mathrm{d}\colon f_*\Omega^p_{X/S}(\mathscr{W}^\vee) \to f_*\Omega^{p+1}_{X/S}(\mathscr{W}^\vee)\big) &\to \mathscr{H} \otimes \mathbf{R}^p f_*\Omega^\bullet_{X/S}(\mathscr{W}^\vee) \\ &\to \mathscr{H} \otimes \mathbf{R}^p f_*^{\mathrm{an}}\mathscr{W}^\vee \otimes \mathscr{O}_S^{\mathrm{an}} \\ &\to \mathscr{O}_S^{\mathrm{an}}. \end{aligned}$$

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We define a *period* of a closed relative p-form α in the Nilsson class of the type considered above to be any multiform function on U of the form $\langle h, \alpha \rangle$, with h a multiform (horizontal) section of $\mathscr{H}$. A period is thus a multiform function of finite determination that is of monodromy subordinate to $\mathscr{H}$. Theorem 1.4 and Theorem II.6.13 thus give us the theorem essentially equivalent to that of (II.7.4).

Theorem 1.8. *Under the hypotheses of (1.7), the periods of a closed relative differential p-form on X in the Nilsson class are functions in the Nilsson class on a suitable dense Zariski open subset of S.*

III.2 The monodromy theorem (by Brieskorn)

The proof of the monodromy theorem given in this section is due to Brieskorn [5].

2.1. Let S be a smooth algebraic curve over $\mathbb{C}$, induced by a smooth projective curve $\overline{S}$ by removing a finite set T of points. For $t \in T$, the *local monodromy group at* t, or the *local fundamental group of* S *at* t, is the fundamental group of $D \setminus \{t\}$, where D is a small disc centred at t. This group is canonically isomorphic to $\mathbb{Z}$, and we call its canonical generator the *monodromy transformation*.

If $\mathscr{V}$ is a local system of $\mathbb{C}$-vector spaces on S, then the local monodromy group at t acts on $V|(D \setminus \{t\})$. If $\mathscr{V}$ is the complexification of a local system of $\mathbb{Z}$-modules of finite type, then the characteristic polynomial of the monodromy transformation has integer coefficients.

Recall that a linear substitution is said to be quasi-unipotent if one of its powers is unipotent. A local system of $\mathbb{C}$-vector spaces on S is said to be *quasi-unipotent* (resp. *unipotent*) *at infinity* if, for all $t \in T$, the corresponding monodromy transformation is quasi-unipotent (resp. unipotent).

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Example 2.2. Let $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$ be the Poincaré half plane, and Γ a torsion-free discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ such that $\Gamma\backslash\mathrm{SL}_2(\mathbb{R})$ is of finite volume. We then know that $\Gamma\backslash X$ is an algebraic curve, with fundamental group Γ. Each finite-dimensional complex representation ρ of Γ thus defines a local system V_ρ on $\Gamma\backslash X$ (and conversely). For V_ρ to be unipotent at infinity, it is necessary and sufficiently for $\rho(\gamma)$ to be unipotent for every element γ of Γ that is unipotent in $\mathrm{SL}_2(\mathbb{R})$.

Theorem 2.3. *Let S be as in (2.1), let i be an integer, and let $f : X \to S$ be a smooth morphism. Suppose that $\mathbf{R}^i f_*\mathbb{C}$ is a local system (i.e. that it is locally constant) (II.6.13). Then $\mathbf{R}^i f_*\mathbb{C}$ is quasi-unipotent at infinity.*

The proof relies on (II.7.4) and on the following theorem of Gelfond ([6] or [2]):

(∗) *If α and $\exp(2\pi i\alpha)$ are algebraic numbers, then α is rational.*

An immediate corollary of (∗) is:

(∗∗) *If N is a matrix with entries in a subfield K of $\mathbb{C}$, and if, for every embedding $\sigma K \to \mathbb{C}$, the characteristic polynomial of $\exp(2\pi i\sigma(N))$ has integer coefficients, then $\exp(2\pi iN)$ is quasi-unipotent.*

Indeed, let α be an eigenvalue of N in an extension K' of K. For every embedding σ of K' into $\mathbb{C}$, $\exp(2\pi i\sigma(\alpha))$ is algebraic. We thus deduce first of all that α is algebraic, since otherwise $\sigma(\alpha)$ could take any non-algebraic value. Then (∗) implies that α is rational, so that the eigenvalues $\exp(2\pi i\alpha)$ of $\exp(2\pi iN)$ are roots of unity.

Proof of Theorem 2.3. By shrinking S if necessary, we can assume that $\mathscr{H} = \mathbf{R}^i f_*\Omega^\bullet_{X/S}$ is locally free.

Let K be a subfield of $\mathbb{C}$ such that f, X, S, $\overline{S}$, and the points of T are all definable over K, i.e. all come from extension of scalars $\sigma_0 : K \to \mathbb{C}$ of p. 127

$$f_0 : X_0 \to S_0 \quad \text{and} \quad T_0 \subset \overline{S}_0(K).$$

The Gauss–Manin connection on $\mathscr{H}_0 = \mathbf{R}^i f_*\Omega^\bullet_{X_0/S_0}$ is regular (II.7.4). There thus exists an extension of $\mathscr{H}_0'$ to a vector bundle on $\overline{S}_0$ such that the connection has at worst a simple pole at each $t \in T_0$. Let N_t be the matrix of the residue of the connection at $t \in T_0$ in a basis of $(\mathscr{H}_0')_t$.

For every embedding $\sigma : K \to \mathbb{C}$, f_0 defines, by extension of scalars,

$$f_{(\sigma)} : X_{(\sigma)} \to S_{(\sigma)}$$

and $\mathscr{H}_{(\sigma)} = \mathbf{R}^i (f_{(\sigma)})_*\Omega^\bullet_{X_{(\sigma)}/S_{(\sigma)}}$ is induced by extension of scalars from $\mathscr{H}_0$. By (II.1.17.1), $\exp(2\pi i\sigma/N_t)$ has the same characteristic polynomial as the local monodromy transformation at t acting on $\mathbf{R}^i(f_{(\sigma)})_*\mathbb{C}$. Thus $\exp(2\pi i\sigma/N_t)$ has a characteristic polynomial with integer coefficients, and, by (∗∗), $\exp(2\pi iN_t)$ is quasi-unipotent, whence Theorem 2.3. □

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