Hodge Theory I

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Translator's note.

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What follows is a translation of the French paper:

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p. 425

We intend to give a heuristic dictionary between statements in l-adic cohomology and statements in Hodge theory. This dictionary has, as its most notable sources sources, [3] and the conjectural theory of Grothendieck motives [2]. Up until now, it has mainly served to formulate conjectures in Hodge theory, and it has sometimes even suggested a proof.

1

Definition 1.1. A *mixed Hodge structure* H consists of

(a) a $\mathbb{Z}$-module $H_{\mathbb{Z}}$ of finite type (the "*integer lattice*");

(b) a finite increasing filtration W of $H_{\mathbb{Q}} = H_{\mathbb{Z}} \otimes \mathbb{Q}$ (the "*weight filtration*");

(c) a finite decreasing filtration F of $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C}$ (the "*Hodge filtration*").

This data is subject to the following condition: there exists a (unique) bigradation of $\mathrm{Gr}_W(H_{\mathbb{C}})$ by subspaces $H^{p,q}$ such that

(i) $\mathrm{Gr}_W^n(H_{\mathbb{C}}) = \bigoplus_{p+q=n} H^{p,q}$

(ii) the filtration F induces on $\mathrm{Gr}_W(H_{\mathbb{C}})$ the filtration

$$\mathrm{Gr}_W(F)^p = \bigoplus_{p' \geqslant p} H^{p',q'}$$

(iii) $\overline{H^{pq}} = H^{qp}$.

[*] https://thosgood.com/translations

A *morphism* $f: H \to H'$ is a homomorphism $f_{\mathbb{Z}}: H_{\mathbb{Z}} \to H'_{\mathbb{Z}}$ such that $f_{\mathbb{Q}}: H_{\mathbb{Q}} \to H'_{\mathbb{Q}}$ and $f_{\mathbb{C}}: H_{\mathbb{C}} \to H'_{\mathbb{C}}$ are compatible with the filtrations W and F (respectively).

The *Hodge numbers* of H are the integers

$$h^{pq} = \dim H^{pq} = h^{qp}. \tag{1.2}$$

We say that H is pure *of weight* n if $h^{pq} = 0$ for $p+q \neq n$ (i.e. if $\mathrm{Gr}^i_W(H) = 0$ for $i \neq n$). We also say that H is a *Hodge structure of weight* n.

The *Tate Hodge structure* $\mathbb{Z}(1)$ is the Hodge structure of weight -2, purely of type $(-1,-1)$, for which $\mathbb{Z}(1)_{\mathbb{C}} = \mathbb{C}$ and $\mathbb{Z}(1)_{\mathbb{Z}} = 2\pi i\mathbb{Z} = \mathrm{Ker}(\exp: \mathbb{C} \to \mathbb{C}^*) \subset \mathbb{C}$. We set $\mathbb{Z}(n) = \mathbb{Z}(1)^{\otimes n}$.

We can show that mixed Hodge structures form an abelian category. If $f: H \to H'$ is a morphism, then $f_{\mathbb{Q}}$ and $f_{\mathbb{C}}$ are strictly compatible with the filtrations W and F (cf. [1, 2.3.5]).

2

Let A be a normal integral ring of finite type over $\mathbb{Z}$, with field of fractions K, and $\overline{K}$ an algebraic closure of K. Let K_{nr} be the largest sub-extension of $\overline{K}$ that is unramified at each prime ideal of A. We know that, or we set, *p. 426*

$$\pi_1(\mathrm{Spec}(A), \overline{K}) = \mathrm{Gal}(K_{nr}/K).$$

For every closed point x of $\mathrm{Spec}(A)$, defined by some maximal ideal m_x of A, the residue field $k_x = A/m_x$ is finite; the point x defines a conjugation class of "Frobenius substitutions" $\varphi_x \in \pi_1(\mathrm{Spec}(A), \overline{K})$. We set $q_x = \# k_x$ and $F_x = \varphi_x^{-1}$.

Let K be a field of finite type over the prime field of characteristic p, let $\overline{K}$ be an algebraic closure of K, let l be a prime number $\neq p$, and let H be a $\mathbb{Z}_l$- (or a $\mathbb{Q}_l$-) module of finite type endowed with a continuous action ρ of $\mathrm{Gal}(\overline{K}/K)$. We will still suppose in what follows that there exists an A as above, with l invertible in A, and such that ρ factors through $\pi_1(\mathrm{Spec}(A), \overline{K}) = \mathrm{Gal}(K_{nr}/K)$. We say that H is *pure of weight* n if, for every closed point x of an non-empty open subset of $\mathrm{Spec}(A)$, the eigenvalues α of F_x acting on H are algebraic integers whose complex conjugates are all of absolute value $|\alpha| = q_x^{n/2}$.

Principle 2.1. If the Galois module H "comes from algebraic geometry", then there exists a (unique) increasing filtration W on $H_{\mathbb{Q}_l} = H \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ (the "*weight filtration*") that is Galois invariant and such that $\mathrm{Gr}^W_n(H)$ is pure of weight n.

We can also further suppose that $\mathrm{Gr}^W_n(H)$ is semi-simple.

When we have a resolution of singularities, we can often give a conjectural definition of W, whose validity follows from the Weil conjectures [5] (cf. §6).

Let μ be the subgroup of $\overline{K}^*$ given by the roots of unity. The *Tate module* $\mathbb{Z}_l(1)$, defined by

$$\mathbb{Z}_l(1) = \mathrm{Hom}(\mathbb{Q}_l/\mathbb{Z}_l, \mu)$$

is pure of weight -2. We set $\mathbb{Z}_l(n) = \mathbb{Z}_l(1)^{\otimes} n$.

It is trivial that every morphism $f: H \to H'$ is strictly compatible with the weight filtration.

Principle 2.1 agrees with the fact that every extension of $\mathbb{G}_m$ ("weight -2") by an abelian variety ("weight $-1 > -2$") is trivial.

3

Translation. The Galois modules that appear in l-adic cohomology have, as analogues, over $\mathbb{C}$, mixed Hodge structures. We further have the dictionary

pure module of weight n	Hodge structure of weight n
weight filtration	weight filtration
Galois-compatible homomorphism	morphism
Tate module $\mathbb{Z}_l(1)$	Tate Hodge structure $\mathbb{Z}(1)$

4

Let X be a complex algebraic variety (i.e. a scheme of finite type over $\mathbb{C}$ that we assume to be separated). Then there exists a subfield K of $\mathbb{C}$, of finite type over $\mathbb{Q}$, such that X can be defined over K (i.e. it comes from an extension of scalars of K to $\mathbb{C}$ applied to a K-scheme X'). Let $\overline{K}$ be the algebraic closure of K in $\mathbb{C}$. The Galois group $\mathrm{Gal}(\overline{K}/K)$ then acts on the l-adic cohomology groups $H^\bullet(X,\mathbb{Z}_l)$; we have

$$H^\bullet(X(\mathbb{C}),\mathbb{Z})\otimes\mathbb{Z}_l = H^\bullet(X,\mathbb{Z}_l) = H^\bullet(X'_{\overline{K}},\mathbb{Z}_l).$$

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By §3, we should expect for the cohomology groups $H^n(X(\mathbb{C}),\mathbb{Z})$ to carry natural mixed Hodge structures. This is what we can prove (see [1, 3.2.5] for the case where X is smooth; the proof is algebraic, using classical Hodge theory [6]). For X projective and smooth, the Weil conjectures imply that $H^n(X,\mathbb{Z}_l)$ is pure of weight n, while classical Hodge theory endows $H^n(X,\mathbb{Z})$ with a Hodge structure of weight n. For every morphism $f\colon X\to Y$, and for K large enough, $f^\bullet\colon H^\bullet(Y,\mathbb{Z}_l)\to H^\bullet(X,\mathbb{Z}_l)$ Galois-commutes (by structure transport); similarly, $f^\bullet\colon H^\bullet(Y,\mathbb{Z})\to H^\bullet(X,\mathbb{Z})$ is a morphism of mixed Hodge structures. For X smooth, the cohomology class Z in $H^{2n}(X,\mathbb{Z}_l(n))$ of an algebraic cycle of codimension n defined over K is Galois invariant, i.e. it defines

$$c(Z)\in\mathrm{Hom}_{\mathrm{Gal}}(\mathbb{Z}_l(-n),H^{2n}(X,\mathbb{Z}_l)).$$

Similarly, the cohomology class $c(Z)\in H^{2n}(X(\mathbb{C}),\mathbb{Z})$ is purely of type (n,n), i.e. it corresponds to

$$c(Z)\in\mathrm{Hom}_{\mathrm{H.M.}}(\mathbb{Z}(-n),H^{2n}(X(\mathbb{C}),\mathbb{Z})).$$

5

If $f\colon H\to H'$ is a Galois-compatible morphism between $\mathbb{Q}_l$-vector spaces of different weights, then $f=0$. Similarly, if $f\colon H\to H'$ is a morphism of pure mixed Hodge structures of different weights, then f is torsion. A more useful remark is

Scholium. Let H and H' be Hodge structures of weight n and n' (respectively), with $n>n'$. Let $f\colon H_\mathbb{Q}\to H'_\mathbb{Q}$ be a homomorphism such that $f\colon H_\mathbb{C}\to H'_\mathbb{C}$ respects F. Then $f=0$.

6

Let X be a smooth projective variety over $\mathbb{C}$, let $D = \sum_1^n D_i$ a normal crossing divisor in X, with D_i all smooth divisors, and let j be the inclusion of $U = X \setminus D$ into X. For $Q \subset [1,n]$, we set $D_q = \bigcap_{i \in Q} D_i$.

In l-adic cohomology, we canonically have

$$R^q j_* \mathbb{Z}_l = \bigoplus_{\#Q=q} \mathbb{Z}_l(-q)_{D_Q} \tag{6.1}$$

and the Leray spectral sequence for j is of the form

$$E_2^{pq} = \bigoplus_{\#Q=q} H^p(D_Q, \mathbb{Q}_l) \otimes \mathbb{Z}_l(-q) \Rightarrow H^{p+q}(U, \mathbb{Q}_l). \tag{6.2}$$

By the Weil conjectures [5], $H^p(D_Q, \mathbb{Q}_l)$ is pure of weight p, so that E_2^{pq} is pure of weight $p+2q$. As a quotient of a sub-object of E_2^{pq}, E_r^{pq} is also pure of weight $p+2q$. By §5, $d_r = 0$ for $r \geqslant 3$, since the weights $p+2q$ and $p+2q-r+2$ of E_r^{pq} and $E_r^{p+q,q-r+1}$ (respectively) are different. Thus $E_3^{pq} = E_\infty^{pq}$. Up to renumbering, the weight filtration of $H^\bullet(U, \mathbb{Q}_l)$ is the abutment of (6.2):

$$\mathrm{Gr}_n^W(H^k(U, \mathbb{Q}_l)) = E_3^{2k-n,n-k}. \tag{6.3}$$

7

In integer cohomology, for the usual topology, the Leray spectral sequence for j is of the form

$${}'E_2^{pq} = \bigoplus_{\#Q=q} H^p(D_Q, \mathbb{Z}) \Rightarrow H^{p+q}(U, \mathbb{Z}). \tag{7.1}$$

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Since each D_Q is a non-singular projective variety, ${}'E_2^{pq}$ is endowed with a Hodge structure of weight p. We set $E_2^{pq} = {}'E_2^{pq} \otimes \mathbb{Z}(-q)$ (a Hodge structure of weight $p+2q$). As an abelian group, $E_2^{pq} = {}'E_2^{pq}$; it is interesting to consider (7.1) as a spectral sequence with initial page E_2^{pq}. By §3, we should expect for $d_2 \colon E_2^{pq} \to E_2^{p+2,q-1}$ to be a morphism of Hodge structures. We prove this by thinking of d_2 as a Gysin morphism. Then E_3^{pq} is endowed with a Hodge structure of weight $p+2q$. By §3, we expect that, *modulo torsion*, the spectral sequence[1] (6.2) degenerates at the E_3 page (i.e. $E_3 = E_\infty$), and that the vanishing of the d_r (for $r \geqslant 3$) is an application of §5. This programme was successfully completed in [1, §3.2]. There, we *define* the weight filtration of $H^\bullet(U, \mathbb{Q})$ as the abutment of (7.1), up to renumbering (6.3).

In fact, to endow the cohomology groups $H^\bullet$ with a mixed Hodge structure, the key point has always been, up until now, to find a spectral sequence E abutting to $H^\bullet$ such that the l-adic analogue of E_2^{pq} be conjecturally pure (of weight $p+2q$); E_2^{pq} should then carry a natural Hodge structure (of weight $p+2q$), and the filtration W is the abutment of E.

[1] *[Trans.] The original refers to (6.4), but this seems to be a typo.*

8

Let Spec(V) be the spectrum of a Henselian discrete valuation ring (a *Henselian trait*) with field of fractions K, and residue field k that is of finite type over the prime field of characteristic p. Let $\overline{K}$ be an algebraic closure of K, and let H be a vector space of finite dimension over $\mathbb{Q}_l$ (for $l \neq p$), on which $\mathrm{Gal}(\overline{K}/K)$ acts continuously. By Grothendieck, we know ([4, Appendix]) that a subgroup of finite index of the inertia group I acts unipotently. By replacing V with a finite extension, we arrive to the case where the action of all of I is unipotent (the *semi-stable* case); it then factors as the largest pro-l-group I_l that is a quotient of I, and canonically isomorphic to $\mathbb{Z}_l(1)$.

Principle 8.1. In the semi-stable case, if the Galois module H "comes from algebraic geometry", then there exists a (unique) increasing filtration W of H (the "*weight filtration*") such that I acts trivially on $\mathrm{Gr}_n^W(H)$, and such that $\mathrm{Gr}_n^W(H)$, as a Galois module under $\mathrm{Gal}(\overline{k}/k) \simeq \mathrm{Gal}(\overline{K}/K)/I$ is pure of weight n.

We can compare this with Principle 2.1 and with the appendix of [4].

If we have a resolution of the singularities, then we can sometimes give a conjectural definition of W, whose validity follows from the Weil conjectures. With the help of the resolution and of Weil, it is sometimes easy to show that, in any case, H splits into pure Galois modules (under $\mathrm{Gal}(\overline{k}/k)$).

Suppose that H is semi-stable. For $T \in I_t$, we define $\log T$ by the *finite* sum $-\sum_{n>0}(\mathrm{Id}-T)^n/n$. The map $(T,x) \mapsto \log T(x)$ can be identified with a homomorphism

$$M\colon \mathbb{Z}_l(1)\otimes H \to H. \tag{8.2}$$

Since $\mathbb{Z}_l(1)$ is of weight -2, we necessarily have (cf. §5)

$$M(\mathbb{Z}_l(1)\otimes W_n(H)) \subset W_{n-2}(H), \tag{8.3}$$

and M induces

$$\mathrm{Gr}(M)\colon \mathbb{Z}_l(1)\otimes \mathrm{Gr}_n^W(H) \to \mathrm{Gr}_{n-2}^W(H). \tag{8.4}$$

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8.5. If X is a non-singular projective variety over an algebraically closed field k_0, then we define

$$L\colon \mathbb{Z}_l(-1)\otimes H^\bullet(X,\mathbb{Z}_l) \to H^\bullet(X,\mathbb{Z}_l)$$

as being the cup product with the cohomology class with a hyperplane section. We note that there is a formal analogy between L and M; in the same way that M is defined by an action of $\mathbb{Z}_l(1)$, we can think of L as being defined by an action of $\mathbb{Z}_l(-1)$; L increases the degree by 2, and $\mathrm{Gr}\,M$ (8.4) decreases it by 2.

9

Let D be the unit disc, $D^* = D \setminus \{0\}$, and X

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{P}^r(\mathbb{C}) \times D \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \mathrm{pr}_2} & \\ & D & \end{array}$$

a family of projective varieties parameterised by D, with f proper, and $f|D^*$ smooth. Keeping the notation of §8, and recalling that, in the analogy between Henselian traits and small neighbourhoods of 0 in the complex line, we have the following dictionary (note that the spectrum of the ring of germs at 0 of holomorphic functions is a Henselian trait):

9.1.

D	$\mathrm{Spec}(V)$
D^*	$\mathrm{Spec}(K)$
a universal covering $\widetilde{D^*}$ of D^*	$\mathrm{Spec}(\overline{K})$
the fundamental group $\pi_1(D^*)$	the inertia group I
(with $\pi_1(D^*) = \mathbb{Z} \simeq \mathbb{Z}(1)_{\mathbb{Z}}$)	(with $I_l = \mathbb{Z}_l(1)$)
X	a projective scheme X over $\mathrm{Spec}(V)$
$X^* = f^{-1}(D^*)$	X_K
$\widetilde{X} = X \times_D \widetilde{D^*}$	$X_{\overline{K}}$
the local system $R^i f_* \mathbb{Z}|D^*$	the Galois module $H^i(X_{\overline{K}}, \mathbb{Z}_l$
$H^i(\widetilde{X}, \mathbb{Z})$	$H^i(X_{\overline{K}}, \mathbb{Z}_l)$

Note that $\widetilde{X}$ is homotopically equivalent to each of the fibres $X_t = f^{-1}(t)$ (for $t \in D^*$): $H^i(X_{\overline{K}}, \mathbb{Z}_l)$ is again analogous to $H^i(X_t, \mathbb{Z})$, and the transformation of the monodromy T corresponds to the action of I.

Here, again, we know that a subgroup of finite index of $\pi_1(D^*)$ acts unipotently on $H^i(\widetilde{X}, \mathbb{Q}) = H^i(X_t, \mathbb{Q})$. We place ourselves in the semi-stable case, where all of $\pi_1(D^*)$ acts unipotently (this reduces to replacing D by a finite covering), and let T be the action of the canonical generator of $\pi_1(D^*)$.

By §3 and §8, we expect for $H^i(\widetilde{X}, \mathbb{Q}) \simeq H^i(X_t, \mathbb{Q})$ to be endowed with an increasing filtration W, for $\mathrm{Gr}^W_n(H^i(\widetilde{X}, \mathbb{Q}))$ to be endowed with a Hodge structure of weight n, for $\log T(W_n) \subset W_{n-2}$, and for $\log T$ to induce a morphism of Hodge structures

$$M_n \colon \mathbb{Z}(-1) \otimes \mathrm{Gr}^W_n(H^i) \to \mathrm{Gr}^W_{n-2}(H^i).$$

We would further like for (8.2), and not just (8.3) and (8.4), to have an analogue.

We have in fact managed to define, for each vector u of the tangent space to D at $\{0\}$, a mixed Hodge structure $\mathscr{H}_u$ on $H^i(\widetilde{X}, \mathbb{Z})$. The filtration W and the Hodge structures on the $\mathrm{Gr}^W_n(H^i)$ are independent of u, and the dependence on u of $\mathscr{H}_u$ can be expressed simply in *p. 430*
terms of T. Analogously to (8.2), we find that, for any u, $\log T$ induces a homomorphism of mixed Hodge structures

$$M \colon \mathbb{Z}(1) \otimes H^i(\widetilde{X}, \mathbb{Z}) \to H^i(\widetilde{X}, \mathbb{Z}).$$

Finally, the analogy in 8.5 is not misleading (but here, the fact that $f|D^*$ is assumed to be proper and smooth is probably essential). We can prove that

$$(\log T)^k \colon \mathrm{Gr}^W_{n+k}(H^n(\widetilde{X}, \mathbb{Q})) \to \mathrm{Gr}^W_{n-k}(H^n(\widetilde{X}, \mathbb{Q}))$$

is an isomorphism for all k (cf. [6, IV 6, Corollary to Theorem 5]). This characterises the filtration W. Up until the present, we have only had an analogue of the positivity theorem of Hodge (cf. [6, IV 7, Corollary to Theorem 7]) in very particular cases. We hope that the mixed structures $\mathscr{H}_u$ determine the asymptotic behaviour, for $t \to 0$, of the family of pure structures $H^i(X, \mathbb{Z})$ (for $t \in D^*$).

References

[1] Deligne, P. Théorie de Hodge. (To appear in *Publ. Math. I.H.E.S.* **40**).

[2] Demazure, M. Motifs des variétés algébriques. *Sém. Bourbaki* (1969–70), Talk no. 365.

[3] Serre, J.-P. Analogues kählériens de certaines conjectures de Weil. *Ann. of Math.* **71** (1960), 392–394.

[4] Serre, J.-P. and Tate, J. Good reduction of abelian varieties. *Ann. of Math.* **88** (1968), 392–517.

[5] Weil, A. Number of solutions of equations in finite fields. *Bull. Amer. Math. Soc.* **55** (1949), 497–508.

[6] Weil, A. *Introduction à l'étude des variétés kählériennes.* Act. Sci. et Ind. **1267**, Hermann (1958).