Summary of essential results in the theory of topological tensor products and nuclear spaces

A. Grothendieck

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Translator's note

This page is a translation into English of the following:

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This article aims to give a summary, without proofs, of the principal results found in my work "Produits tensoriels topologiques et espaces nucléaires," which will be published in the *Memoirs of the Amer. Math. Society* (and which I will refer to as [PTT]). The main concern throughout [PTT] was that of being exhaustive, both in terms of studying all the questions raised by the topics covered, as well as trying to state the more difficult results as theorems that were as general as possible. This work was also very dense, and the important simple ideas risked being sometimes hidden behind technical details. This is why this bowdlerised summary is possibly useful in giving a more assimilable outline of the theory. Some extra comments, interesting but not necessary for the general understanding of this summary, as well as some hints for certain proofs, have been placed between stars, ⋆ like this ⋆.

The importance of topological tensor products shows itself in many different settings: p. 74

a. The notion of the topological tensor product forms the foundations of a simple and general formulation of *Fredholm theory*, including, alongside the classical case of an integral operator defined by a continuous kernel, many other operators that are defined in the most important functional spaces.[1]

b. The many variants of the notion of topological tensor product give rise, by duality, to the definition of many remarkable classes of bilinear forms and linear operators, whose study is only just barely covered in [PTT, chap. I, §4]. In particular, the techniques introduced there, conveniently systematised and exploited, allow us to obtain entirely unexpected results in the *theory of linear transformations between the spaces L^1, L^2, and L^∞*, and their topological-vectorial analogues (these results being, as of yet, not definitive, and thus unpublished). I might return to this subject, and restrict myself to explaining, in a rather different way, the systematic work of von Neumann–Schatten on the remarkable classes of compact operators in a Hilbert space [8].

c. From the point of view of this current work, the most important application of topological tensor products is the theory of *nuclear spaces*. We explain this theory, generalise it, and make precise the famous "theory of kernels" of L. Schwartz, and further discover new properties, even for the most classical of spaces. Here, the topological tensor calculus is the most simple, since the majority of variants of the notion of topological tensor product coincide, and their properties thus sum. For now, there are not many applications of the general theorems that we obtain to specific theories. The most interesting seems to be a topological-vectorial variant of the "Künneth theorem," giving the homology of a complex defined as the tensor product of two complexes, a variant which seems useful in topological algebra.

d. Generally, it seems to me that the notions of topological tensor product are perfect for giving a suggestive and manageable *language* that would be good to use in many

[1] Such a formulation of Fredholm theory seems to have appeared for the first time in A. Ruston, "Direct product of Banach spaces and linear functional equations," *Proc. of the London Math. Soc.* **3** (1951), 1. My work on this subject was conceived independently of his (in the autumn of 1951), and is rather different.

situations in functional analysis, especially since we have theorems (some of which are non-trivial) at our disposition from which we can benefit. I hope that this summary (or, better, [PTT]) will succeed in giving the reader a similar impression, before the publication of the articles promised above.

Terminology and notation

Generally we follow the terminology and notation of [3], apart from the fact that we call the semi-reflexive spaces of [3] *reflexive*. We only consider, unless otherwise stated, spaces that are *locally convex* and *separated*; by "quotient space" of a space E, we mean the quotient of E by a *closed* vector subspace. The *dual* of E, written E', is assumed to be, unless otherwise stated, endowed with the strong topology (i.e. the topology of bounded convergence). The dual of E', or the *bidual* of E, written E'', is assumed to be, unless otherwise *p. 75* stated, endowed with the topology given by uniform convergence on the equicontinuous subsets of E', which induces the original topology on E. We will eventually need to appeal to certain notions defined and studied in [6], most notably that of a $(\mathscr{DF})$*-space*. For our purposes here, it will suffice to know that the dual of a $(\mathscr{F})$-space is a $(\mathscr{DF})$-space; that every normed space is a $(\mathscr{DF})$-space; and that the dual of a $(\mathscr{DF})$-space is an $(\mathscr{F})$-space.

Let E, F, and G be locally convex spaces. Denote by $\mathrm{B}(E,F;G)$ (resp. $\mathscr{B}(E,F;G)$) the space of continuous bilinear maps (resp. of separately continuous bilinear maps, i.e. linear with respect to each variable) from $E \times F$ to G. Denote by $\mathrm{L}(E;F)$ the space of continuous linear maps from E to F. Denote by $\mathscr{B}_e(E'_s,F'_s)$ the space of separately continuous bilinear forms on the product of the weak duals E'_s and F'_s of E and F, endowed with the *biequicontinuous topology*, i.e. the topology given by uniform convergence on the products of an equicontinuous subset of E' with equicontinuous subset of F'. This space is complete if and only if the spaces E and F are complete.

We define a *bounded* (resp. *compact*, resp. *weakly compact*) *linear map* from E to F to be a linear map from E to F that sends a suitable neighbourhood of 0 to a bounded (resp. relatively compact, resp. relatively weakly compact) subset of F.

For short[2], if E is a vector space, we say *disk* or *disked set in E* to mean a convex and circled (a.k.a. balanced) subset of E. If E is a locally convex space, and A a bounded disk in E, then we denote by E_A the vector space generated by A, and endowed with the norm $\|x\|_A = \inf_{x\in\lambda A} |\lambda|$. If A is closed, then the unit ball of E_A is A. If A is complete, then E_A is complete. If V is a disked neighbourhood of 0 in E, then E_V denotes the normed space given by passing to the quotient under the semi-norm $\|x\|_V = \inf_{x\in\lambda V} |\lambda|$.

Recall that a locally convex space is said to be *quasi-complete* if its closed bounded subsets are complete, *barrelled* (resp. *quasi-barrelled*) if the bounded subsets of its weak dual (resp. of its strong dual) are equicontinuous, and *bornological* if every set of linear forms on E that are uniformly bounded on every bounded subset is equicontinuous. If E is quasi-complete, then barrelled is equivalent to quasi-barrelled; in any case, bornological implies quasi-barrelled.

[2]This terminology was suggested to me by R.E. Edwards. *[Trans.] The seemingly more popular terminology these days is to say "absolutely convex" instead of "disked," and to speak of the "absolutely convex hull" instead of the "disked hull."*

1 Topological tensor products

1.1 Generalities on $E\hat{\otimes}F$

[PTT, chap. 1, §1, no. 1 and no. 3]

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The axiomatic definition of the algebraic tensor product $E\otimes F$ of two vector spaces E and F, and of the canonical bilinear map $(x,y)\mapsto x\otimes y$ from $E\times F$ to $E\otimes F$ ([1]) asks only that, for every vector space G, the bilinear maps from $E\times F$ to G correspond bijectively with linear maps f from $E\otimes F$ to G, where the map corresponding to f is given by $(x,y)\mapsto f(x\otimes y)$.

Theorem 1. *If E and F are locally convex spaces, then we can endow $E\otimes F$ with a locally convex topology such that, for every locally convex space G, the* continuous *bilinear maps from $E\times F$ to G correspond exactly to* continuous *linear maps from $E\otimes F$ to G. Further, such a topology is unique.*

Then the *equicontinuous* subsets of $\mathrm{B}(E,F;G)$ correspond exactly to the *equicontinuous* subsets of $\mathrm{L}(E\otimes F;G)$ as well. Unless otherwise mentioned, $E\otimes F$ is assumed to be endowed with the above topology, called the *projective tensor product of the topologies of E and F*; endowed with this topology, $E\otimes F$ is called the *projective topological tensor product* of E and F.

If E and F are normed, then $E\otimes F$ is normable, and we can even find a norm such that, for every *normed* space G, the above isomorphism between $\mathrm{B}(E,F;G)$ and $\mathrm{L}(E\otimes F;G)$ preserves the natural norms. Further, such a norm is unique. This norm on $E\otimes F$, denoted by $u\mapsto \|u\|$, where the norms of E and F are implicit, is the lower bound of the quantities $\sum_i \|x_i\|\|y_i\|$ over all representations of u in the form $u=\sum_i x_i\otimes y_i$ (and this norm has already been considered, in [8]). It is also the gauge of the set $\Gamma(U\otimes V)$, where U (resp. V) is the unit ball in E (resp. F), and where $U\otimes V$ denotes the set of $x\otimes y$ such that $x\in U$ and $y\in V$ (with Γ denoting, as per usual, the balanced hull). In the case where E and F *p. 77* are general locally convex spaces, a fundamental system of neighbourhoods of 0 in $E\otimes F$ is obtained by taking the sets $\Gamma(U\otimes V)$, where U (resp. V) runs over a fundamental system of neighbourhoods of 0 in E (resp. F).

We can introduce the completion of $E\otimes F$, denoted by $E\hat{\otimes}F$, and called the *completed projective tensor product* of E and F. If E and F are normed spaces, then $E\hat{\otimes}F$ is a Banach space (with a well-defined norm!). If E and F are metrisable, then $E\hat{\otimes}F$ is of type $(\mathscr{F})$. We have, by definition, the *scholium*: if E and F are locally convex spaces, and G is a *complete* locally convex space, then the continuous bilinear maps from $E\times F$ to G correspond bijectively to the continuous linear maps from $E\hat{\otimes}F$ to G.

This claim still holds true for equicontinuous sets of maps. In particular, the dual of $E\hat{\otimes}F$ is $\mathrm{B}(E,F)$, with a correspondence between the equicontinuous subsets (which already suffices to characterise the induced topology on $E\otimes F$).

I do not know if, when E and F are of type $(\mathscr{F})$, this algebraic isomorphism from the dual of $E\hat{\otimes}F$ to $\mathrm{B}(E,F)$ is a *topological* isomorphism, when we endow $\mathrm{B}(E,F)$ with the topology given by bibounded convergence, i.e. uniform convergence on the products of two bounded sets ("the problem of topologies"). An equivalent question is the following: is every bounded subset of $E\hat{\otimes}F$ contained in the closed disked hull of a set $A\hat{\otimes}B$, where A (resp. B) is a bounded subset of E (resp. F)?

⋆ We now give some general tips for calculations involving $E\hat{\otimes}F$ ([PTT, chap. 1, §1, no. 3]). If $E = \prod_i E_i$ and $F = \prod_j F_j$ (as topological-vectorial products), then $E\hat{\otimes}F$ can be identified with $\prod_{i,j} E_i\hat{\otimes}F_j$. If $E = \sum_i E_i$ (as a topological direct sum), and if F is a normable space, then $E\hat{\otimes}F$ can be identified with the topological direct sum $\sum_i(E_i\hat{\otimes}F)$. This remains true if F is an arbitrary $(\mathscr{DF})$ space, provided that I is countable, and these statements can also be generalised to the case where E is the inductive limit (in the most general sense) of a family E_i of spaces. If E and F are both of type $(\mathscr{F})$ (resp. type $(\mathscr{DF})$), then so too is $E\hat{\otimes}F$. Similarly, if E and F are quasi-normable spaces, or Schwartz spaces (see definitions in [6, Section 3]), then so too is $E\hat{\otimes}F$. ⋆

1.2 The space $E\hat{\otimes}F$ when E and F are of type $(\mathscr{F})$

[PTT, chap. 1, §2, no. 1] *p. 78*

Theorem 2. *Let E and F be $(\mathscr{F})$ spaces. Then every element of $E\hat{\otimes}F$ is the sum of an absolutely convergent series of the form*

$$u = \sum_i \lambda_i x_i \otimes y_i$$

where (x_i) (resp. (y_i)) is a bounded sequence in E (resp. F), and where (λ_i) is a summable sequence of scalars.

(It is also true that, if (x_i), (y_i), and (λ_i) are given as above, then the series $\sum_i \lambda_i x_i \otimes y_i$ is always absolutely convergent in $E\hat{\otimes}F$, and so we have a *characterisation* of the elements of $E\hat{\otimes}F$). If E and F are normed, then we can suppose in the above that $\|x_i\| \leqslant 1$, $\|y_i\| \leqslant 1$, $\sum_i |\lambda_i| \leqslant \|u\|_1 + \varepsilon$, where $\varepsilon > 0$ is arbitrary and given in advance. In these two statements, if u runs over a compact subset of $E\hat{\otimes}F$, then we can suppose that the sequences (x_i) and (y_i) remain fixed (and we can even suppose that they are sequences that tend to 0), and that (λ_i) runs over a compact subset of ℓ^1 (the space of summable sequences). We have an analogous statement for the concrete representation of convergent sequences in $E\hat{\otimes}F$. Theorem 2 and its previous variants serve mainly as a way to:

{.itenv title="Corollary" latex="{Corollary}"} Let E and F be spaces of type $(\mathscr{F})$. Then every compact subset K of $E\hat{\otimes}F$ is contained inside the canonical image of the unit ball of a space $E_A\hat{\otimes}F_B$, where A (resp. B) is a compact disked subset of E (resp. F). A fortiori, K is contained in the closed convex hull of $A \otimes B$. :::

This fact also implies that, on B(E,F), the "bicompact convergence" topology is identical to the compact convergence topology on the dual of $E\hat{\otimes}F$.

⋆ For the proof of Theorem 2, we suppose, for simplicity, that E and F are Banach spaces, and let I be the product of their unit balls, and $i \mapsto x_i$ and $i \mapsto y_i$ the projections from I to its factors. It is easy t see that the linear map $(\lambda_i) \mapsto \sum_i \lambda_i x_i \otimes y_i$ from $\ell^1(I)$ to $E\hat{\otimes}F$ is a metric homomorphism from the former to a dense subspace of the latter, and thus *onto* the latter, whence it indeed follows that $E\hat{\otimes}F$ can be identified with a quotient space of $\ell^1(I)$. *p. 79*

From Theorem 2, we can extract results of the following type: let $\mathscr{G}$ be a locally compact group (resp. a Lie group), then every summable function f (resp. every infinitely differentiable function of compact support) on $\mathscr{G}$ is of the form $\sum \lambda_i g_i * h_i$, where $(\lambda_i) \in \ell^1$, and where (g_i) and (h_i) are bounded sequences in $\mathrm{L}^i(\mathscr{G})$ (resp. in $\mathscr{D}(\mathscr{G})$, the space of infinitely differentiable functions of compact support on $\mathscr{G}$); we thus immediately conclude that f is a linear combination of functions of positive type that are in $\mathrm{L}(\mathscr{G})$ (resp. $\mathscr{D}(\mathscr{G})$). In the former case, we can also restrict to functions that are all of compact support (with the supports of g_i and h_i being contained inside a compact subset that depends only on the compact support of f). There is a simple direct proof in the case of $\mathrm{L}^1(\mathscr{G})$, but I do not think that there is one for $\mathscr{D}(\mathscr{G})$, where the question presents difficulties even for $\mathscr{G} = \mathbb{R}$ (by using the Fourier transformation). ⋆

1.3 Calculation of $\mathrm{L}^1 \hat{\otimes} E$

[PTT, chap. 1, §2, no. 2]

Let M be a locally compact space endowed with a measure $\mu \geqslant 0$, E a Banach space, and $\mathrm{L}^1_E(\mu)$ the space of μ-integrable maps from M to E ([2]) endowed with its usual norm: $\|f\|_1 = \int \|f(t)\| \mathrm{d}\mu(t)$. We denote by $\mathrm{L}^1(\mu)$ the space of μ-summable scalar functions. Then there exists an obvious bilinear map $(\varphi, a) \mapsto \varphi \cdot a$ from $\mathrm{L}^1(\mu) \times E$ to $\mathrm{L}^1_E(\mu)$ that is of norm $\leqslant 1$, and thus defines a linear map of norm $\leqslant 1$ from $\mathrm{L}^1(\mu) \hat{\otimes} E$ to $\mathrm{L}^1_E(\mu)$.

Theorem 3. *The above map from $\mathrm{L}^1(\mu) \hat{\otimes} E$ to $\mathrm{L}^1_E(\mu)$ is a metric isomorphism from the former space to the latter.*

To see this, we can immediately reduce to the case where E is of finite dimension, and then proceed by transposition. It then suffices to apply the classical theorem of Dunford–Pettis that characterises continuous linear maps from $\mathrm{L}^1(\mu)$ to E'.

If E is an arbitrary locally convex space, then we denote by $\mathrm{L}^1_E(\mu)$ the completion of the separated space associated to the space of continuous maps with compact support from M to E, endowed with its family of semi-norms $f \mapsto \int p(f(t)) \mathrm{d}\mu(t)$ (where p runs over a fundamental family of continuous semi-norms on E). Then Theorem 3 easily implies that $\mathrm{L}^1_E(\mu)$ is again isomorphic to $\mathrm{L}^1(\mu) \hat{\otimes} E$.

Corollary. *If E is a closed vector subspace of the Banach space F, then the canonical linear map from $\mathrm{L}^1(\mu) \hat{\otimes} E$ to $\mathrm{L}^1(\mu) \hat{\otimes} F$ is a metric isomorphism.*

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This recovers, for example, the well-known fact that every continuous linear map from E to the dual $\mathrm{L}^\infty(\mu)$ of $\mathrm{L}^1(\mu)$ can be extended to a linear map of the same norm from F to $\mathrm{L}^\infty(\mu)$; or, dually, that every continuous linear map from $\mathrm{L}^1(\mu)$ to a quotient space F'/E^0 of a Banach dual by a weakly closed vector subspace comes from a linear map of the same norm from $\mathrm{L}^1(\mu)$ to F'.

⋆ The analogue of Theorem 3 for L^p spaces is false for all $p > 1$. However, Theorem 3 applies, in an essential way, in many important places in the theory outlined here. We now give some less important applications (see [PTT, chap. 1, §2, no. 2] for details). In Theorem 3, if we take $E = c_0$ (the space of scalar sequences that converge to 0), and note that $\mathrm{L}^1_E(\mu)$ can then be identified with the space of latticially bounded sequences in $\mathrm{L}^1(\mu)$ that converge to 0 almost everywhere, then we see that such sequences in $\mathrm{L}^1(\mu)$ form a category of invariant sequences from the topological-vectorial point of view. In particular, a continuous linear map from $\mathrm{L}^1(\mu)$ to a space $\mathrm{L}^1(\nu)$ sends latticially bounded sequences that converge to 0 almost everywhere to sequences of the same type. We thus also see that the latticially bounded subsets of $\mathrm{L}^1(\mu)$ form an invariant category from the topological-vectorial point of view. If we take $F = \ell^p$, with $1 \leqslant p < +\infty$, then we similarly obtain a topological-vectorial interpretation of the sequences (f_i) in $\mathrm{L}^1(\mu)$ such that

$$\int \left(\sum_i |f_i(t)|^p \right)^{1/p} \mathrm{d}\mu(t) < +\infty.$$

Such sequences are sent to sequences of the same type by any continuous linear map from $\mathrm{L}^1(\mu)$ to a space $\mathrm{L}^1(\nu)$. Another interesting application of Theorem 3 is the following: every bounded subset M of $\mathrm{L}^1(\mu)\hat{\otimes}E$ is contained in the canonical image of the unit ball of a space $\mathrm{L}^1(\mu)\hat{\otimes}E_A$, where A is a closed bounded disk in the space E of type $(\mathscr{F})$; a fortiori, M is contained in the closed disked hull of $B \otimes A$, where B is the unit ball of $\mathrm{L}^1(\mu)$, which solves the "problem of topologies" described in §1.1. ⋆

1.4 Other examples

If H is a Hilbert space, then the elements of $H'\hat{\otimes}H$, identified with endomorphisms of H (the *Fredholm maps* or *nuclear maps* from H to H — see **TO-DO**) are exactly the endomorphisms u such that the positive Hermitian operator $\sqrt{u^*u}$ is compact and has a summable sequence of eigenvalues, and $\|u\|_1$ is then equal to the sum of the eigenvalues of $\sqrt{u^*u}$ (repeated with multiplicities, of course). We obtain the operators that have *p. 81* already been studied in [4] and [8]. In fact, u is also a Fredholm operator if and only if its Hermitian components $\frac{1}{2}(u+u^*)$ and $\frac{1}{2i}(u-u^*)$ are, i.e. if they are compact Hermitian operators whose sequences of eigenvalues are summable. *Relation to Hilbert–Schmidt operators: If A and B are Hilbert–Schmidt operators, then AB is a Fredholm operator, and* $\|AB\|_1 \leqslant \|A\|_2\|B\|_2$; and, conversely, every Fredholm operator u is the product of two Hilbert–Schmidt operators A and B of norm $\|A\|_2 = \|B\|_2 = \sqrt{\|u\|_1}$. All of these facts are elementary (one we know the spectral decomposition of compact Hermitian operators to a Hilbert space) and well known.

Numerous other examples of products $E\hat{\otimes}F$, relating to *nuclear spaces*, will be seen in **TO-DO**.

⋆ In the setting of infinite-dimensional Banach spaces, I do now know, even in particular cases, other concrete characterisations of the elements of $E\hat{\otimes}F$ apart from those that we have given. Also, the elements of $c_0\hat{\otimes}E$ (where E can be any complete locally convex space) can be identified with *certain* sequences in E that converge to 0, that we can call *nuclearly convergent to* 0; but if E is an infinite-dimensional Banach space, then we always obtain a strictly smaller class than the class of all sequences converging to 0 (see **TO-DO**). We even show that (if E is an infinite-dimensional Banach space), for any sequence (λ_i) of positive scalars that is not square-summable, there exists a sequence (x_i) in E that does not converge nuclearly to 0, and such that $\|x_i\| = \lambda_i$ for all i. However, if, for example, E is the space $C(K)$ of continuous functions on a compact space, then we show that every square-summable sequence in $C(K)$ converges nuclearly to 0. We also point out that, in any complete locally convex space, every summable sequence converges nuclearly to 0. ⋆

1.5 $E\hat{\hat{\otimes}}F$ spaces

[PTT, chap. 1, §3, no. 3]

If E and F are Banach spaces, then $E \otimes F$ can be considered as a vector subspace of the Banach space $\mathrm{B}(E',F')$ of continuous bilinear forms on $E' \times F'$. The completion of $E \otimes F$ *p. 82* under the norm induced by $\mathrm{B}(E',F')$ is denoted by $E \hat{\hat{\otimes}} F$, which is thus a complete normed vector subspace of $\mathrm{B}(E',F')$. Every reasonable normed topology on $E \otimes F$ is included between the topology induced by $E \hat{\otimes} F$ and the topology induced by $E \hat{\hat{\otimes}} F$. If E and F are now both arbitrary locally convex spaces, then we can again consider $E \otimes F$ is a space of bilinear forms on $E' \times F'$, and endow it with the biequicontinuous-convergence topology (i.e. the topology given by uniform convergence on the products of an equicontinuous subset of E' with an equicontinuous subset of F'): the completion of $E \otimes F$ under this topology is again denoted by $E \hat{\hat{\otimes}} F$. If E and F are complete, then the space $\mathscr{L}_e(E'_s,F'_s)$ of separately continuous bilinear forms on the product $E'_s \times F'_s$ of the weak duals E'_s and F'_s endowed with the biequicontinuous-convergence topology is complete, and so $E \hat{\hat{\otimes}} F$ can be identified with a topological vector subspace of $\mathscr{L}_e(E'_s,F'_s)$. Then the elements of $E \hat{\hat{\otimes}} F$ can be identified with certain separately weakly continuous bilinear maps on $E' \times F'$, or even with certain weakly continuous linear maps from E' to F; these linear maps send equicontinuous subsets of E' to relatively compact subsets of F, and the converse is true in all known cases (see **TO-DO**).

The topology on $E \otimes F$ induced by $E \hat{\otimes} F$ is finer than that induced by $E \hat{\hat{\otimes}} F$, whence we have a canonical continuous linear map

$$E \hat{\otimes} F \to E \hat{\hat{\otimes}} F.$$

An important, unsolved, problem is the question of whether or not this map is always bijective (see **TO-DO**). We note that it seems extremely plausible that, if E and F are Banach spaces such that the above map $E \hat{\otimes} F \to E \hat{\hat{\otimes}} F$ is a topological isomorphism (or even simply a topological homomorphism, i.e. here a map from the first space *onto* the second), then E or F is of finite dimension. This is true if, for example, E contains a vector subspace isomorphic to an ℓ^p space or to c_0.

Let E and F be Banach spaces. The norm induced by the dual of $E \hat{\otimes} F$ on $E' \otimes F'$ is clearly the norm induced by $E' \hat{\hat{\otimes}} F'$. On the other hand, the norm induced on $E' \otimes F'$ by the dual of $E \hat{\hat{\otimes}} F$ is, in all known cases (see **TO-DO**), identical to the norm induced by $E' \hat{\otimes} F'$. This duality, which barely appears in the present summary, is a precious tool for various questions (e.g. [PTT, chap. 1, §4, no. 6]). The following theorem (which is, truth be told, *p. 83* trivial), can be seen as the dual response to Theorem 3.

Theorem 4. *Let M be a locally compact space, let $\mathrm{C}_0(M)$ the space of continuous scalar functions on M that are "zero at infinity," endowed with the uniform convergence norm, and let E be a locally convex complete space. Then $\mathrm{C}_0(M) \hat{\hat{\otimes}} E$ is canonically isomorphic to the space $\mathrm{C}_0(M,E)$ of continuous maps from M to E that are zero at infinity, endowed with the uniform convergence topology (and with the uniform norm if E is a Banach space). In particular, $c_0 \hat{\hat{\otimes}} E$ is isomorphic to the space of sequences in E that tend to* 0.

As with the spaces $C(K) \hat{\otimes} E$ below, here the spaces $\mathrm{L}'(\mu) \hat{\hat{\otimes}} E$ do not, in general, have a special interpretation as functional spaces. We note, however, that *the space $\ell^1 \hat{\hat{\otimes}} E$ can be understood as the space of summable sequences in E* (i.e. of "commutatively convergent" sequences in E).

1.6 Tensor product of linear maps

[PTT, chap. 1, §1, no. 2]

Let E_i and F_i (for $i = 1,2$) be locally convex spaces, and let u_i be a continuous linear map from E_i to F_i. In algebra, we define a linear map $u_1 \otimes u_2$ from $E_1 \otimes E_2$ to $F_1 \otimes F_2$ by the formula $(u_1 \otimes u_2)(x_1 \otimes x_2) = u_1 x_1 \otimes u_2 x_2$. This map is continuous if we endow $E_1 \otimes E_2$ and $F_1 \otimes F_2$ with the topologies induced by $E_1 \hat{\otimes} E_2$ and $F_1 \hat{\otimes} F_2$ (resp. by $E_1 \hat{\hat{\otimes}} E_2$ and $F_1 \hat{\hat{\otimes}} F_2$). It then follows that $u_1 \otimes u_2$ extends, by continuity, to a continuous linear map $u_1 \hat{\otimes} u_2$ from $E_1 \hat{\otimes} E_2$ to $F_1 \hat{\otimes} F_2$ (resp. $u_1 \hat{\hat{\otimes}} u_2$ from $E_1 \hat{\hat{\otimes}} E_2$ to $F_1 \hat{\hat{\otimes}} F_2$). These maps will also simply be denoted by $u_1 \otimes u_2$ whenever there is no fear of confusion.

Theorem 5. *If each u_i is a topological homomorphism from E_i to a dense subspace of F_i, then $u_1 \hat{\otimes} u_2$ is a topological homomorphism from $E_1 \hat{\otimes} E_2$ to a dense subspace of $F_1 \hat{\otimes} F_2$. If each u_i is a topological isomorphism from E_i to F_i, then $u_1 \hat{\hat{\otimes}} u_2$ is a topological isomorphism from $E_1 \hat{\hat{\otimes}} E_2$ to $F_1 \hat{\hat{\otimes}} F_2$.*

These claims remain true if the E_i and F_i are normed and we consider *metric* homomorphisms and *metric* isomorphisms. Particularly interesting is the following corollary, which can also be obtained by an application of Theorem 2.

Corollary. *If the E_i and F_i are $(\mathscr{F})$-spaces (for $i = 1,2$), and the u_i are homomorphisms from E_i* onto *F_i, then $u_1 \hat{\otimes} u_2$ is a homomorphism from $E_1 \hat{\otimes} E_2$* onto *$F_1 \hat{\otimes} F_2$.* p. 84

As particular cases of this corollary, we obtain interesting lifting properties of vector functions with values in a quotient space of an $(\mathscr{F})$-space E. If e.g. f is an infinitely differentiable map from an open subset of $\mathbb{R}^n$ to E/F, then it comes from an infinitely differentiable map from this same open subset to E. There is an analogous result for holomorphic functions, or infinitely differentiable functions on $\mathbb{R}^n$ that decay rapidly, or summable functions of a certain measure, etc.[3] Another application is the following: let D be a differential operator in the space $\mathscr{E}(U)$ of infinitely differentiable functions on an open subset U of $\mathbb{R}^n$, let E be an $(\mathscr{F})$-space, and let $\mathscr{E}(U,E)$ be the space of infinitely differentiable maps from U to E. Then we have $\mathscr{E}(U,E) = \mathscr{E}(U) \hat{\otimes} E = \mathscr{E}(U) \hat{\hat{\otimes}} E$ (see §2.5). Let D_E be the operator in $\mathscr{E}(U,E)$ defined by D, and then $D_E = D \hat{\otimes} 1$, where 1 is the identity on E. Then, if D is a topological homomorphism (resp. an *onto* topological homomorphism), then so too is D_E. Indeed, in the case of an onto homomorphism, this is a particular case of the corollary of Theorem 5, and, in the general case, we use Theorem 5 as well as the fact that, for every quotient space F of $\mathscr{E}(U)$, we have $F \hat{\otimes} E = F \hat{\hat{\otimes}} E$ (since $\mathscr{E}(U)$, and thus F, are nuclear; see Definition 1 in §2.2 and Theorem 3 in §2.3).

We note that, if u_1 and u_2 are topological isomorphisms, then $u_1 \hat{\otimes} u_2$ is not, in general, a topological isomorphism (nor is, in general $u_1 \hat{\hat{\otimes}} u_2$ a topological homomorphism, if u_1

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[3] With respect to this point, we done that we can prove, by a completely different method, the following claim, which can be thought of as dual to the corollary of Theorem 3: *Let M be a locally compact and paracompact space (e.g. a compact space), and f a continuous map from M to a quotient space E/F of an $(\mathscr{F})$-space E. Then f comes from a continuous map from M to E.* Since the space $\mathscr{E}^{(m)}(V)$ of m-times continuously differentiable functions on a infinitely differentiable paracompact manifold V is isomorphic to a direct factor of a space of the form $C(M)$ (as I noted in "Sur les applications linéaires faiblement compacts d'espaces du type $C(K)$," *Can. J. Math.* **5** (1953), p. 144), it thus follows that the analogous lifting theorem holds also for m-times continuously differentiable maps from V to E/F.

and u_2 are *onto* topological homomorphisms). If each E_i is identified with a topological-vectorial subspace of F_i by u_i, then the canonical map $u_1\hat{\otimes}u_2$ from $E_1\hat{\otimes}E_2$ to $F_1\hat{\otimes}F_2$ is a topological isomorphism if and only if every equicontinuous set of bilinear forms on $E_1 \times E_2$ is the set of restrictions of an equicontinuous set of bilinear forms on $F_1 \times F_2$. If F_1 and F_2 are of type $(\mathscr{F})$, then it suffices to consider sets consisting of *one single* bilinear form. In general, this criterion will not hold true, but it is linked to an existence problem of topological complements. *p. 85*

⋆ More precisely, if E_1 and E_2 are direct factors, then $E_1\hat{\otimes}E_2$ can be identified with a topological-vectorial subspace of $F_1\hat{\otimes}F_2$. Also, if E is a topological-vectorial subspace of the Banach space F, and if the canonical map $E\hat{\otimes}G \to F\hat{\otimes}G$ is a topological isomorphism when $G = F'$, then E'' is a direct factor of F''; thus, in the frequent case where E is already a direct factor of E'', E will also be a direct factor of F.

A useful case where the tensor product $u\hat{\otimes}v$ of two topological isomorphisms is a topological isomorphism is the following: *If E'' is the bidual of E, then $E\hat{\otimes}F$ can be identified with a topological-vectorial subspace (resp. a normed vector subspace, if E and F are normed) of $E''\hat{\otimes}F$.* ⋆

1.7 Nuclear maps

[PTT, chap. 1, §3, no. 2]

Let E and F be Banach spaces. The continuous bilinear map $(x',y) \mapsto x' \otimes y$ from $E' \times F$ to $\mathrm{L}(E,F)$ defines a natural continuous linear map from $E'\hat{\otimes}F$ to $\mathrm{L}(E,F)$; the elements of the image of $E'\hat{\otimes}F$ in $\mathrm{L}(E,F)$ are said to be *nuclear maps* from E to F. (We also define, between arbitrary locally convex spaces, the notions of *trace maps* and *Fredholm maps* — see Appendix 1 — which coincide with the notion of nuclear maps if E and F are Banach spaces; in this case, we can thus freely switch between saying "nuclear maps," "trace maps," or "Fredholm maps.") Nuclear maps from E to F form a vector space which can be identified with a quotient space of $E'\hat{\otimes}F$ (and with $E'\hat{\otimes}F$ itself in all known cases — see "problem of bijectivity" in §1). The quotient norm, again denoted by $u \mapsto \|u\|_1$, is called the *trace norm* of the nuclear operator u.

If E and F are arbitrary locally convex spaces, then a linear map u from E to F is said to be *nuclear* if it is the composition of a sequence of three operators

$$E \xrightarrow{\alpha} E_1 \xrightarrow{\beta} F_1 \xrightarrow{\gamma} F$$

where E_1 and F_1 are Banach spaces, β is a nuclear map from E_1 to F_1, and α and γ are continuous linear maps. It is equivalent to say that there exists an weakly closed, equicontinuous, disked subset A of E', and a bounded disked subset B of F such that F_B is complete, and finally some $u_0 \in E'_A\hat{\otimes}F_B$ such that u is the operator from E to F defined by u_0 (*a priori*, u_0 defines a nuclear map from $\widehat{E}_{A^0}$ to F_B). *A nuclear map is always compact* (i.e. sends a suitable neighbourhood of 0 to a relatively compact set). Even better: by the corollary of Theorem 2, we can suppose (in the above) that B and A are *compact* subsets of F and strong E' (respectively). Theorem 2, applied directly, also gives the following: *nuclear maps from E to F are exactly the maps that are sums of series (always absolutely convergent in* $\mathrm{L}(E,F)$ *endowed with the bounded-convergence topology) $u = \sum \lambda_i x'_i \otimes y_i$, where (x'_i) is an equicontinuous sequence in E', (y_i) is a* ***TO-DO*** *a compact disk of F, and (λ_i) is a summable sequence of scalars*. If we compose a nuclear map, on the *p. 86* left or on the right, with a continuous linear map, then we obtain another nuclear map. The dual of a nuclear map from E to F is a nuclear map from strong F' to strong E' (and even from F' endowed with the uniform convergence on compact disks of F, to strong E').

Nuclear maps from E to itself (more precisely, the slightly larger category of Fredholm maps from E to itself) form a natural domain for *Fredholm theory*. Here, our interest lies in other properties of these operators, that result directly from either Theorem 2 or the corollary of Theorem 5.

Theorem 6.

Let E and G be locally convex spaces, and F a vector subspace of E. Then:

a. *Every nuclear map from F to G is the restriction of a nuclear map from E to G.*

b. *Suppose that F is closed, and that every compact disc of E/F is contained in the canonical image of a bounded disc A of E such that E_A is complete (for example, a* complete *bounded disc). (It suffices, for example, for E to be an $(\mathscr{F})$-space, or for it be the dual of an $(\mathscr{F})$-space and for F to be* weakly *closed.) Then every nuclear map from G to E/F can be obtained from a nuclear map from G to E by passing to the quotient.*

We have analogous statements for "equinuclear" sets of maps, if by that we mean a set of maps from a locally convex space M to another locally convex space N, contained in the set of maps defined by the unit ball of a space $M'_A\hat{\otimes}N_B$, where A is a weakly closed equicontinuous disk of M', and B a bounded disk in N such that N_B is complete. *p. 87*

We note that, if u is a nuclear map from E to F, M a closed vector subspace of E contained in the kernel of u, and N a closed vector subspace of F containing $u(E)$, then, in general, the linear map from E/M to F or from E to N defined by *u is not nuclear*, even if E and F are reflexive Banach spaces. However, if M (resp. N) admits a topological complement, then the map from E/M to F (resp. from E to N) defined by u will itself be nuclear. This is the case, in particular, if E (resp. F) is a Hilbert space.

1.8 Integral linear maps, integral bilinear forms

[PTT, chap. 1, §4, no. 3 and no. 4]

Theorem 7.

Let E and F be locally convex (resp. normed) spaces, and v a separately continuous bilinear form on $E\times F$. Then the following conditions are equivalent:

a. *$u\mapsto\langle u,v\rangle$ is a linear form on $E\otimes F$ that is continuous for the topology induced by $E\hat{\hat{\otimes}}F$ (resp. $u\mapsto\langle u,v\rangle$ is a linear form on $E\otimes F$ of norm $\leqslant 1$ when $E\otimes F$ is endowed with the norm induced by $E\hat{\hat{\otimes}}F$).*

b. *v is contained in the closed disked hull in $\mathscr{B}_s(E,F)$ (the space $\mathscr{B}(E,F)$ endowed with the simple-convergence topology) of a set $M\otimes N$, where M is an equicontinuous subset of E', and N an equicontinuous subset of F' (resp. M the unit ball of E', and N the unit ball of F').*

c. *There exists a measure μ on the product space of a weakly compact equicontinuous subspace M of E' with a weakly compact equicontinuous subspace N of F' (resp. a measure μ of norm $\leqslant 1$ on the product of the unit ball of E' with the unit ball of F', endowed with the product of the weak topologies) such that we have the formula*

$$v = \int_{M\times N} x' \otimes y' \mathrm{d}\mu(x', y')$$

(the weak integral in $\mathscr{B}(E,F)$, in duality with $E \otimes F$).

d. *There exists a compact space endowed with a positive measure μ of norm $\leqslant 1$, a continuous linear map α from E to $\mathrm{L}^\infty(\mu)$, and a continuous linear map β from F to $\mathrm{L}^\infty(\mu)$ (resp. the same, but with α and β also of norm $\leqslant 1$) such that we have $u(x,y) = \langle \alpha x, \beta y\rangle$ for $x \in E$, $y \in F$.*

p. 88

A bilinear form on $E \times F$ is said to be *integral* if it satisfies any of the equivalent conditions of Theorem 7. In particular, the dual of $E \hat{\hat{\otimes}} F$ can be identified with the space $\mathrm{J}(E,F)$ of integral bilinear forms on $E \times F$. If E and F are Banach spaces, then $\mathrm{J}(E,F)$ will be endowed with the dual norm of the Banach space $E \hat{\hat{\otimes}} F$, which we call the *integral norm*, and denote by $\|v\|_1'$. Similarly, a linear map v from one locally convex space E to another G is said to be *integral* if the corresponding bilinear form on $E \times G'$ is integral.

If E and G are Banach spaces, then we also call the integral norm of the corresponding bilinear form of v the integral norm of v.

Recall that, in all known cases, when E and F are Banach, the natural linear map from $E' \hat{\otimes} F'$ to $\mathrm{J}(E,F)$ is a metric isomorphism from the first space to the second (see §1.5), which is why we use the notation $\|v\|_1'$ for the integral norm, closely related to the notation $\|v\|_1$ for the trace norm. Criterion (d) of Theorem 7 takes the following form (which we state for Banach spaces, for simplicity) for integral linear maps: *Let v be a linear map from one Banach space E to another F. Then v is integral and of integral norm $\leqslant 1$ if and only if the map from E to F'' that it defines can be obtained by composing a linear map of norm $\leqslant$ from E to some space $\mathrm{L}^\infty(\mu)$, constructed with a suitable positive measure of norm $\leqslant 1$ on a compact space, with the identity map from $\mathrm{L}^\infty(\mu)$ to $\mathrm{L}^1(\mu)$, and finally with a linear map of norm $\leqslant$ from $\mathrm{L}^1(\mu)$ to F''.* Similarly, criterion (b) of Theorem 7 easily gives: *The linear map v from E to F is integral and of integral norm $\leqslant 1$ if and only if it is an adherent point in $\mathrm{L}_s(E,F_s)$ (where F_s denotes F endowed with the weak topology, and $\mathrm{L}_s(E,F_s)$ denotes $\mathrm{L}(E,F_s)$ endowed with the simple-convergence topology) of the disked hull of the set of the $x' \otimes y$, where x' (resp. y) runs over the unit ball of E' (resp. of F); or if it is adherent to the set of nuclear operators of trace-norm $\leqslant 1$.*

Examples. Let E and F be arbitrary locally convex spaces. Then every bilinear form on $E \times F$ defined by an element of a space $E'_A \hat{\otimes} F'_B$, where A (resp. B) is a weakly closed disked equicontinuous subset of E' (resp. of F'), is integral. Thus every nuclear map from E to F is integral. The converse is false, even for Banach spaces, since the identity p. 89 map from $\mathrm{L}^\infty(\mu)$ to $\mathrm{L}^1(\mu)$ is integral, but it is not, in general, even compact. If $E = \mathrm{C}(M)$ (resp. $F = \mathrm{C}(N)$) is the space of continuous scalar functions on the compact space M (resp. N), then we have seen (Theorem 4) that $\mathrm{C}(M) \hat{\otimes} \mathrm{C}(N)$ can be identified with its norm to the space $\mathrm{C}(M \times N)$, and so the space of integral bilinear forms on $\mathrm{C}(M) \times \mathrm{C}(N)$ can be identified with its norm to the space of Radon measures on the compact space $M \times N$. Other examples will be seen in §1.9.

By composing an integral linear map on the left or on the right with a continuous linear map, we obtain another integral linear map. The transpose of an integral map from E to F is an integral map from strong F' to strong E'.

Using, for example, criterion (a) of Theorem 7, we see that, if E_1 and E_2 are locally convex spaces, and F_1 (resp. F_2) is a topological vector subspace of E_1 (resp. of E_2), then every integral bilinear form on $F_1 \times F_2$ can be extended to a integral bilinear form on $E_1 \times E_2$, with equal integral norm if E_1 and E_2 are normed (compare with Theorem 6). The most important properties of integral maps are summarised in the following:

Theorem 8.

Let u be an integral linear map from a locally convex space E to a locally convex space F.

1. *If F is quasi-complete, then u is weakly compact, and sends weakly compact subsets of E to compact subsets of F. If v is a linear map from F to a locally convex space G that sends bounded subsets to weakly relatively compact subsets, then $v \circ u$ is a compact map.*

2. *Let v be a linear map from F to an $(\mathscr{F})$-space G that sends bounded subsets to weakly relatively compact subsets (resp. a linear map from a $(\mathscr{DF})$-space G to E that sends bounded subsets to weakly relatively compact subsets of E). Then $v \circ u$ (resp. $u \circ v$) is a* nuclear *map from E to G (resp. from G to F''). If E, F, and G are Banach spaces, then $\|v \circ u\|_1' \leqslant \|v\| \|u\|'$.*

Corollary 1. *The composition of two integral maps is a nuclear map.*

Other corollaries. An integral map from E to F is nuclear if F is a reflexive $(\mathscr{F})$-space, and it is a nuclear map from E to F'' if E is a reflexive $(\mathscr{DF})$-space. An integral map from E to F sends summable sequences to absolutely summable sequences, and weakly convergent sequences to nuclearly convergent sequences (see the end of §1.4). *p. 90*

⋆ So, considering the unit circumference T of the complex plane, endowed with its Haar measure μ, suppose that the sequence (a_n) in the set $\mathbb{Z}$ of integers is such that $(\varepsilon_n a_n)$ is the sequence of coefficients of a function in $\mathrm{L}^\infty(\mu)$, for some sequence (ε) of numbers equal to $+1$ or -1; then we can easily see that the sequence $a_n z^n \in \mathrm{L}^\infty(\mu)$ is summable, and is thus an absolutely summable sequence in $\mathrm{L}^1(\mu)$, whence, immediately, $(a_n) \in \ell^1$. We have thus obtained an analogue of the well-known theorem of Littlewood (in which L^∞ is replaced by L^1, and ℓ^1 by ℓ^2). We note that Littlewood's theorem can be obtained in the same way, as a consequence of the following theorem, which has much more general consequences (and which will later be published, along with some of its various consequences): *every summable sequence in an $\mathrm{L}^1(\mu)$-space (for an arbitrary measure) has a sequence of norms that is square summable* (and even belonging to $\ell^2 \hat{\otimes} \mathrm{L}^1(\mu)$; this is dual to the theorem, noted at the end of §1.4, that says that every square summable sequence in a $\mathrm{C}(K)$-space — and even every sequence belonging to $\ell^2 \hat{\hat{\otimes}} \mathrm{C}(K)$ — is nuclearly convergent to 0).

We will give some hints concerning the proof of Theorem 8, that rely in an essential manner on criterion (d) of Theorem 7. The fact that u is a weakly compact map follows immediately from the fact that the identified map from $\mathrm{L}^\infty(\mu)$ to $\mathrm{L}^1(\mu)$ is a weakly compact map. The other claims of the theorem follow easily from the second part, which is more difficult. We can reduce to showing that every weakly compact linear map from $\mathrm{L}^1(\mu)$ to an $(\mathscr{F})$-space G induces a nuclear map from $\mathrm{L}^\infty(\mu)$ to G, and thus (Theorem 3) comes from some $f \in \mathrm{L}^1_G(\mu)$. But a theorem of Dunford–Pettis–Phillips tells us that such a map is in fact given by a *strongly measurable and bounded* map f from M to G. Note that Corollary 1, which is important due to its application in the theory of nuclear spaces, admits a simpler direct proof: we can reduce to showing that, if μ and ν are measures on compact M and N, then a continuous linear map u from $\mathrm{L}^1(\mu)$ to $\mathrm{L}^\infty(\nu)$ defines a *nuclear* map from $\mathrm{L}^\infty(\mu)$ to $\mathrm{L}^1(\nu)$. But u can be identified with a continuous linear form on $\mathrm{L}^1(\mu)\hat{\otimes}\mathrm{L}^1(\nu)$, which is isomorphic to the space $\mathrm{L}^1(\mu\otimes\nu)$ (Theorem 3), and so u is defined by a measurable and bounded function f on $M\times N$, which can be identified a fortiori with an element of $\mathrm{L}^1(\mu\otimes\nu) = \mathrm{L}^1(\mu)\hat{\otimes}\mathrm{L}^1(\nu)$. This latter element indeed defines a nuclear map from $\mathrm{L}^\infty(\mu)$ to $\mathrm{L}^1(\nu)$, which happens to be exactly the map induced by u. QED. *p. 91*

In PTT, we deduce Theorem 8 from more general results (see [PTT, chap. 1, §4, no. 2]). The majority of [PTT, chap. 1, §4] (the densest part of the whole work) is dedicated to the exposition of these results and their many consequences, which we cannot give in this summary. ⋆

1.9 Integral linear maps to an L^1 space or a $\mathrm{C}_0(M)$ space

[PTT, chap. 1, §4, no. 4]

⋆ We can characterise the integral linear from a locally convex space E to an $\mathrm{L}^1(\mu)$ space (where μ is an arbitrary measure on a locally compact space M): they are the linear maps that send a suitable neighbourhood V of 0 in E to a *latticially bounded* subset of $\mathrm{L}^1(\mu)$. If E is normed, with V its unit ball, and $h = \sup_{x\in V}|ux|$ (so that h is a positive element of $\mathrm{L}^1(\mu)$), then $\|u\|'_1 = \|h\|_1$. If E or $\mathrm{L}^1(\mu)$ is separable, then the theorem of Dunford–Pettis gives an equivalent criterion (which we state, as an example, in the case where E is assumed to be normed): *there exists a weakly measurable map f from M to E' such that $\|f(t)\|$ is a summable function of t, and such that, for all $x\in E$, ux is the class in $\mathrm{L}^1(\mu)$ of the function $t\mapsto\langle x, f(t)\rangle$; then $\|u\|'_1 \leqslant \int\|f(t)\|\mathrm{d}\mu(t)$ (and we have equality for a suitable choice of f).* We can also characterise the *nuclear* maps from a locally convex space E to $\mathrm{L}^1(\mu)$: they are those which send a suitable neighbourhood V of 0 in E to a subset A of $\mathrm{L}^1(\mu)$ which is latticially bounded and further *equimeasurable* (by which we mean that, for every compact $K\subset M$, and every $\varepsilon>0$, there exists a compact K_0 such that $\mu(K\cap K_0^c)\leqslant\varepsilon$, and such that the $\varphi\in A$ agree almost everywhere on K_0 with the functions from an equicontinuous and uniformly bounded set of functions on K_0). If we suppose, for simplicity, that E is a Banach space, then this is equivalent to saying that the map in question is given by an *integrable* [?] map f from M to E' (and, indeed, it follows immediately from Theorem 3 that this indeed implies that u is nuclear). *p. 92*

Dually, let M be a locally compact space, and E a Banach space (for simplicity), and suppose that either M is metrisable and countable at infinity, or that E is separable. Then the integral maps u from $\mathrm{C}_0(M)$ (the space of continuous functions on M that are "zero at infinity") to E', i.e. the continuous linear forms on $\mathrm{C}_0(M,E) = \mathrm{C}_0(M)\hat{\check{\otimes}}E$, are exactly those given by a measure μ on M and a weakly μ-measurable and bounded map f from M to E' by the formula $u\varphi = \int\varphi(t)f(t)\mathrm{d}\mu(t)$. We can further suppose that $\|f(t)\| = 1$ for all t, and that $\|\mu\|_1 = \|u\|'_1$. Using Theorem 3, we also find that the *nuclear* maps from $C_0(M)$ to E', or even from $C_0(M)$ to an arbitrary Banach space F, are exactly those given by a pair (u, f) as above, but with f being an *integrable* map from M to F; it is no longer necessary here to make any separability hypotheses. In particular, if K and L are compact spaces, μ a measure on K, and $N(x,y)$ a continuous scalar function on $K \times L$, then the map $f \mapsto \int f(x)N(x,y)\mathrm{d}\mu(x)$ from $C(K)$ to $C(L)$, defined by the continuous kernel N, is nuclear. Note that we have just described two remarkable categories of "vectorial measures" on M, which we call *integral vectorial measures* and *nuclear vectorial measures* (respectively) on M.

The fact that we can characterise the integral and nuclear maps from a Banach space (for example) E to $L^1(\mu)$ by properties concerning their images of the unit ball is completely special to $L^1(\mu)$ (see the end of §1.7) and is also linked to the corollary of Theorem 3. Similarly, we can show that the integral linear maps from $L^1(\mu)$ to a locally convex space E can be characterised by properties concerning their images of the unit ball of $L^1(\mu)$, a feature that we will not further study here (see [PTT, chap. 1, §4, no. 6]). ⋆

Appendix 1: Various variants of the notion of topological tensor product

[PTT, chap. 1, §3]

On $E \otimes F$, we can introduce a large number of distinct interesting topologies (even when E and F are Banach spaces). We will content ourselves here with noting the existence of a unique locally convex topology on $E \otimes F$ such that, for every locally convex space G, *separately continuous* bilinear maps from $E \times F$ to G correspond exactly to *continuous* linear maps from $E \otimes F$ to G. To separately equicontinuous sets of bilinear maps from $E \times F$ then correspond equicontinuous sets of linear maps from $E \otimes F$. Endowed with this topology, $E \otimes F$ is called the *inductive tensor product* of E and F, and its completion, denoted by $E \bar{\otimes} F$. Its dual is thus the space $\mathscr{B}(E,F)$, and the equicontinuous subsets of this dual are the separately equicontinuous sets of bilinear forms on $E \times F$ (which already suffice to determine the topology of the inductive tensor product). The inductive tensor product topology on $E \otimes F$ is finer than the projective tensor product topology, and these two topologies are identical if and only if the separately equicontinuous sets of bilinear forms on $E \times F$ are already equicontinuous (for example, if E and F are of type $(\mathscr{F})$, or if E and F are of type $(\mathscr{DF})$ and barrelled). So if E is a non-normable space, then the two topologies above on $E \otimes E'$ give distinct duals (since the canonical bilinear form on $E \times E'$ is separately continuous but not continuous). — If E is the inductive limit (in the general sense) of a family of spaces E_i, and F the inductive limit of a family of spaces F_j, then the topological-vectorial subspace H of $E \bar{\otimes} F$ generated by the canonical images of the spaces $E_i \bar{\otimes} F_j$ is the inductive limit of these latter spaces (whence the name "inductive" tensor product). Unfortunately, the space H above is often not complete, i.e. it is distinct from $E \bar{\otimes} F$. — The analogous statement of the above, for when E and F are *product* spaces, is only true under fairly restrictive conditions, e.g. if F is the topological-vectorial product of a family of spaces *of type* $(\mathscr{F})$. — Finally, note that the notion of topological tensor product that we have just developed gives rise to a notion of tensor product of two continuous linear maps, completely analogous to the notion developed in §1.6. p. 93

We should define a remarkable dense subset of $E \bar{\otimes} F$, which is more important that $E \bar{\otimes} F$ itself (even though it is often not complete, since it is distinct from $E \bar{\otimes} F$): it is

the subspace given by the union of the canonical images of the spaces $E_A \bar{\otimes} F_B = E_A \hat{\otimes} F_B$, where A (resp. B) is a bounded disk of E (resp. F) such that E_A (resp. F_B) is complete. The elements of this subspace are called *Fredholm kernels* in $E \bar{\otimes} F$, and they appear, for example, in the theory of nuclear spaces (see Corollary 3 of Theorem 1 in §2.2). The subspaces of Fredholm kernels are sent to one another by tensor products of continuous linear maps. — Theorem 2 (in §1.2) shows that, if E and F are of type $(\mathscr{F})$, then every element of $E \bar{\otimes} F = E \hat{\otimes} F$ is already a Fredholm kernel; it further gives an explicit structure theorem for Fredholm kernels in the general case: Fredholm kernels in $E \bar{\otimes} F$ can be represented *p. 94*
by series

$$u = \sum \lambda_i x_i \otimes y_i$$

where (x_i) (resp. (y_i)) is a series extracted from a compact disk of E (resp. F), and where (λ_i) is a summable sequence of scalars. Thus u itself comes from an element of some space $E_A \hat{\otimes} F_B$, where A and B are *compact* disks.

We define a *Fredholm map* from E to F to be a map defined by a Fredholm kernel of $E' \bar{\otimes} F$. Such a map is weakly continuous, but not necessarily continuous; but if every strongly compact disk of E' is equicontinuous (in particular, if the topology on E is the Mackey topology $\tau(E,E')$), then every Fredholm map from E to F is already nuclear, and a fortiori continuous. — Note that it is the Fredholm kernels of $E' \bar{\otimes} E$ that form the natural domain for Fredholm theory.

Let E be a locally convex space, and take a locally convex topology on its dual that is less fine than the weak topology. Then the canonical bilinear form on $E \times E'$ is separately continuous, and thus defines a continuous linear form on $E \bar{\otimes} E'$, called the *trace* form. Let F be another locally convex space; let $A \in \mathscr{B}(E,F)$, and suppose that the corresponding linear map ${}^tA$ from F to E' is continuous (which will be the case, for example, if we take the strong topology on E' and assume that F is barrelled); then $1 \otimes {}^tA$ is a continuous linear map from $E \bar{\otimes} F$ to $E \bar{\otimes} E'$, also denoted by $u \mapsto {}^tAu$. Then the passage to the trivial limit gives, for all $u \in E \bar{\otimes} F$,

$$\langle u, A \rangle = \operatorname{Tr} {}^tAu.$$

This makes the duality between $E \bar{\otimes} F$ and $\mathscr{B}(E,F)$ explicit by means of the trace form. This formula immediately generalises for the natural pairing corresponding to any "reasonable" type of completed topological tensor product (PTT, §3, no. 3, prop. 17). — If K is a compact space endowed with a measure μ, then we have already seen (cf. §1.9) that the integral operator defined by a continuous kernel $N(x,y)$ (defined on $K \times K$) is a nuclear operator in $\mathrm{C}(K)$. We can easily show that its trace is exactly $\int N(x,x) \mathrm{d}\mu(x)$.

Appendix 2: The properties and problems of approximation

[PTT, chap. 1, §5]

The most important problem that remains to be solved in the theory of topological tensor products is the following "*bijectivity problem*": is the canonical map from $E \hat{\otimes} F$ to *p. 95*
$E \hat{\hat{\otimes}} F$ always bijective? By the Hahn–Banach theorem, this problem can be turned into of the variations of the "*approximation problem*": is every continuous bilinear form on $E \times F$ the limit, in the weak topology defined by $E \hat{\otimes} F$, of degenerate continuous bilinear forms, i.e. of those coming from $E' \otimes F'$? Under this form of the problem, we see that we can reduce to the case where E and F are Banach spaces. Since then the bicompact-convergence topology (i.e. uniform convergence on products of a compact of E with a compact of F)

on $\mathrm{B}(E,F)$ gives the dual $E\hat{\otimes}F$ (cf. §1.2, Theorem 2), we can replace the weak topology defined by $E\hat{\otimes}F$ in the "approximation problem" with the bicompact-convergence topology. This also proves that we can assume E and F to be separable. By continuing with such procedures (notably the systematic use of Theorem 2), we have given, in [PTT, chap. 1, §5, prop. 37], a large number of other equivalent formulations of the above conjecture. It suffices, for example, to assume in the above that E is a topological-vectorial subspace of c_0, and that F is its dual. It even suffices to prove that, if $u \in E'\hat{\otimes}E$ defines a zero nuclear operator, then $\mathrm{Tr}\,u = 0$. A more concrete formulation of this latter statement is the following: let $u = (u_{ij})$ be a matrix that represents an element of $\ell^1\hat{\otimes}c_0$ (i.e. such that $\sum_i \sup_j |u_{ij}| < +\infty$), and such that $u^2 = 0$; then $\mathrm{Tr}\,u = \sum u_{ii} = 0$. Another formulation: let $K(x,y)$ be a continuous kernel on $X \times X$ (where X is a compact space endowed with a positive measure μ) such that $K \circ K = 0$; then $\mathrm{Tr}K = \int K(x,x)\mathrm{d}\mu(x) = 0$. Or: let $f(x,y)$ be a continuous function on the product of two compact spaces X and Y; then f is the uniform limit of linear combinations of functions of the type $f(x,b)f(a,y)$. In these latter two examples, it suffices to do the proof in only *one* case for which the general conjecture has been proven, provided that μ is not the sum of a sequence of discrete masses in the first case, or that X and Y are infinite in the second case. Other formulations are given in what follows.

We say that a locally convex space E satisfies the *approximation condition* if the identity map from E to itself is the limit, in the topology given by uniform convergence on every precompact subset, of continuous linear maps of finite rank. Then, for *every* locally p. 96 convex space F, *every* continuous linear map from E to F, or from F to E, is the limit, in the precompact-convergence topology, of continuous linear maps of finite rank. If E is a Banach space, then this also implies that, for every Banach space F, every compact linear map from F to E is the limit, in the sense of the *norm*, of continuous linear maps of finite rank; or even that, for every Banach space G, the space $G\hat{\hat{\otimes}}E$ is identical to the space of compact and weakly continuous linear maps from G' to E. We can show, by using Theorem 2, that this is equivalent to the fact that, for every Banach space F, the canonical map from $E\hat{\otimes}F$ to $\mathrm{B}(E',F')$ is bijective; and, in this condition, it suffices to take $F = E'$ (and we thus have here a *bijectivity condition*), and even to suppose that the trace of any $u \in E'\hat{\otimes}E$ that defines a zero operator is zero. — We can show that the dual E' of a Banach space E satisfies the approximation condition if and only if every compact linear map from E to a Banach space F is the limit, in the sense of the norm, of continuous linear maps of finite rank, and that this implies that E itself satisfies the approximation condition. — The "approximation problem" described above can also be stated as follows: does every locally convex space satisfy the approximation condition? We have already seen that we can restrict to closed vector subspaces of c_0.

The spaces L^p (for $1 \leqslant p \leqslant +\infty$), constructed from an arbitrary measure, the spaces $\mathrm{C}(K)$ (of continuous functions on a compact space K), as well as the duals, biduals, etc. of these spaces, all satisfy the approximation condition (and even the stronger "metric approximation condition"; see below). Nuclear spaces satisfy the approximation condition (see §2), and so too, more generally, do spaces that are isomorphic to subspaces of products of Hilbert spaces (these types of spaces arising rather frequently in practice). I give other examples in [PTT, chap. 1, §5, no. 3], including, notably, the most important amongst the Banach spaces given by distributions over $\mathbb{R}^n$ (essentially those that are between $(\mathscr{D})$ and $(\mathscr{D}')$ and that are stable under translation and under multiplication by $\varphi \in (\mathscr{D})$).

We say that a Banach space E satisfies the *metric approximation condition* if the identity map from E to itself is the limit, uniform on every compact subset, of linear applications of finite rank *and of norm* $\leqslant 1$. This is a metric strengthening of the approximation condition, and we can give analogous reformulations ([PTT, chap. 1, §5, no. 2]). We note the following: for every Banach space F, the canonical map from $E\hat{\otimes}F$ to the space $\mathrm{J}(E',F')$ of integral bilinear forms on $E'\times F'$ is a metric isomorphism. It again suffices to prove that the canonical map from $E\hat{\otimes}E'$ to $\mathrm{J}(E',E)$ is a metric isomorphism. We do not know of any Banach space that does not satisfy the metric approximation condition. p. 97

We can again prove that E' satisfies the metric approximation condition if and only if, for every Banach space F, the canonical map from $E'\hat{\otimes}F'$ to the space $\mathrm{J}(E,F)$ of integral bilinear forms on $E\times F$ is a metric isomorphism; we can then prove that E then also satisfies the metric approximation condition. More profound is the following result:

Theorem 9. *Let E, F, and G be Banach spaces, and u (resp. v) a continuous linear map from E to F (resp. from F to G). Suppose that one of the maps (u or v) is weakly compact, and that the other is the uniform limit, on every compact subset, of continuous linear maps of finite rank. Then* $w = v\circ u$ *is the uniform limit, on every compact subset, of continuous linear maps of finite rank* and of norm $\leqslant \|w\|$.

Corollary. *Let E be a reflexive Banach space. For E to satisfy the metric approximation condition, it suffices for it to satisfy the approximation condition.*

These statements give conclusions of a metric nature from purely topological hypotheses, and thus hold true if we replace the given norms by equivalent norms. In this respect, the corollary to Theorem 9 gives, even for a Hilbert space, a new approximation result.

2 Nuclear spaces

2.1 Introduction to nuclear spaces

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In the majority of examples where E is a given concrete complex locally convex space (a space of functions, or distributions, for example), and F is an arbitrary complete locally convex space, we do not know how to concretely, simultaneously characterise $E\hat{\otimes}F$, $E\hat{\hat{\otimes}}F$, and its topological-vectorial extension $\mathscr{B}_e(E'_s,F'_s)$ (for example, interpret these spaces as spaces of functions or distributions with vector values, characterised in a simple way). But we do often know how to concretely describe a locally convex space P sitting between $E\hat{\otimes}F$ and $E\hat{\hat{\otimes}}F$, or at least between $E\hat{\otimes}F$ and $\mathscr{B}_e(E'_s,F'_s)$ ("sitting between" meaning that the maps $E\hat{\otimes}F\to P$ and $P\to\mathscr{B}_e(E'_s,F'_s)$ are continuous).

Example 1. Let $E=\mathrm{L}^p(\mu)$, where $1\leqslant p<+\infty$; if F is a Banach space (and, by extension, if F is an aribtrary complex locally convex space), the we define the space $\mathrm{L}^p_F(\mu)$ of "p-th power integrable" maps from the measured space M to F (cf. [2]) in a natural way, and we easily see that

$$\mathrm{L}^p\hat{\otimes}F\subset\mathrm{L}^p_F\subset\mathrm{L}^p\hat{\hat{\otimes}}F.$$

In general, L^p_F is different from $\mathrm{L}^p\hat{\otimes}F$ and $\mathrm{L}^p\hat{\hat{\otimes}}F$, and the elements of $\mathrm{L}^p\hat{\otimes}F$ do not admit an obvious internal characterisation as maps from M to F (unless $p=1$, in which case see

§1, Theorem 3), whereas the elements of $\mathrm{L}^p \hat{\hat{\otimes}} F$ can not even be interpreted, in general, as measurable or scalar-measurable maps from M to F.

Example 2. For spaces E consisting of actual functions (and not only of classes of functions, like L^p), we can often characterise $\mathscr{B}_e(E'_s, F'_s)$, thanks to the following:

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Lemma. *Let E be a function space on a set M, endowed with a locally convex topology that is finer than the simple-convergence topology. Then, for every complete locally convex space F, we can interpret $\mathscr{B}_e(E'_s, F'_s)$ as the space of maps f from M to F such that, for all $y' \in F'$, the function $f_{y'}(t) = \langle f(t), y' \rangle$ belongs to E (with f belonging* ***TO-DO: scalairement*** *to E), and such that $f_{y'}$ runs over a weakly relatively compact subset of E as y' runs over an equicontinuous subset of F'. (This second condition is excessive if E is reflexive and of type $(\mathscr{F})$ or type (**TO-DO**)).*

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