Notes on the Riemann ζ function, 2

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Translator's note.

This text is one of a series[*] of translations of various papers into English. The translator takes full responsibility for any errors introduced in the passage from one language to another, and claims no rights to any of the mathematical content herein.

What follows is a translation of the French paper:

Balazard, M., Saias, E., and Yor, M. "Notes sur la fonction ζ de Riemann, 2". *Advances in Mathematics*, Volume **143** (1999), 284–287.

p. 284

We denote by $\sum_{\Re\rho>1/2}$ a sum over the possible zeros of $\zeta(s)$ with real part greater than $\frac{1}{2}$, where the zeros of multiplicity m are counted m times. The goal of this note is the proof of the following result.

Theorem. *We have*

$$\frac{1}{2\pi}\int_{\Re(s)=1/2}\frac{\log|\zeta(s)|}{|s|^2}|\mathrm{d}s| = \sum_{\Re\rho>1/2}\log\left|\frac{\rho}{1-\rho}\right|. \tag{1}$$

In particular, the Riemann hypothesis is true if and only if

$$\int_{\Re(s)=1/2}\frac{\log|\zeta(s)|}{|s|^2}|\mathrm{d}s| = 0.$$

Proof. This proof consists of two steps.

First step. We start by stating some properties satisfied by a generic function f in the Hardy space $H^p(\mathbf{D})$, where $\mathbf{D} = \{z \in \mathbb{C} : |z| < 1\}$, and p is a positive real number. We denote by f^* the function defined almost everywhere on the trigonometric circle $\partial\mathbf{D} = \{z \in \mathbb{C} : |z| = 1\}$ by $f^*(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$. We use the letter z to denote an element of the trigonometric disc $\mathbf{D}$, and write

$$s = s(z) = \frac{1}{2} + \frac{1+z}{2(1-z)} = \frac{1}{1-z}.$$

This formula defines a conformal representation of the disc $\mathbf{D}$ in the semi-plane $\Re(s) > 1/2$. *p. 285*

[*] https://thosgood.com/translations

By Jensen's formula (see, for example, [4, Theorem 3.61]), we have, for $f(0) \neq 0$ and $r < 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})|\,\mathrm{d}\theta = \log|f(0)| + \sum_{\substack{|\alpha|<r \\ f(\alpha)=0}} \log\frac{r}{|\alpha|} \tag{2}$$

where, in the sum, the zeros of multiplicity m are counted m times. Denote by

$$\exp\left\{-\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,\mathrm{d}\mu(\theta)\right\}$$

the singular interior factor of f. As r tends to 1, Equation (2) becomes (cf. [2, p. 68])

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})|\,\mathrm{d}\theta = \log|f(0)| + \sum_{\substack{|\alpha|<1 \\ f(\alpha)=0}} \log\frac{1}{|\alpha|} + \int_{-\pi}^{\pi} \mathrm{d}\mu(\theta). \tag{3}$$

This formula is a consequence of the factorisation theorem for functions in H^p; it is stated in [2] for $p = 1$, but also holds for all positive values of p.

Second step. Now consider the function

$$f(z) = (s-1)\zeta(s)$$

(where $s = 1/(1-z)$). The elementary properties of the Riemann ζ function (see, for example, [5]) allow us to show that, on one hand, f belongs to the Hardy space $H^{1/3}(\mathbf{D})$, and, on the other hand, that the measure μ associated to the singular interior factor of f is zero (for this latter point, it suffices to reuse the argument developed by Bercovici and Foias for the interior factor of the functions $(\theta-\theta^s)\zeta(s)(s+1/2)/s$, found in the proof of [1, Proposition 2.1]). We can equally show that

$$\int_{-\pi}^{\pi} \log|f^*(e^{i\theta})|\,\mathrm{d}\theta = \int_{\Re(s)=1/2} \frac{\log|\zeta(s)|}{|s|^2}\,|\mathrm{d}s|,$$

$$\log|f(0)| = 0,$$

$$\sum_{\substack{|\alpha|<1 \\ f(\alpha)=0}} \log\frac{1}{|\alpha|} = \sum_{\Re\rho>1/2} \log\left|\frac{\rho}{1-\rho}\right|.$$

With all this information, our result follows from Equation (3). p. 286

□

We finish with some remarks. There are statements related to ours in the works [7, 6] of Wang and Volchkov. It is even possible that Jensen himself was aware of Equation (1) (the reader can consult the article [3] where Jensen informs Mittag–Leffler of his discovery of Equation (2)). It seem interesting, however, to present things as we have done here, and this is for the following three reasons:

(a) Equation (1) is simpler than those that appear in [7, 6];

(b) we show here that, to establish Equation (1), it is natural to place ourselves in the framework of Hardy spaces;

(c) the form of the integral in Equation (1) allows us to interpret this result via Brownian motion, as we show below.

Denote by $Z = X + iY$ the planar Brownian motion from 0 (or from 1), and by $Z_{T_{1/2}} = \frac{1}{2} + iY_{T_{1/2}}$ its first point of impact on the critical line $\Re s = 1/2$, where $T_{1/2} := \inf\{t : X_t = 1/2\}$. We know that $Y_{T_{1/2}}$ follows a Cauchy law with parameter 1/2. In other words, the law of $Y_{T_{1/2}}$ has density $1/2\pi(1/4 + t^2)$. Thus the second part of the theorem can be stated in the following manner: the Riemann hypothesis is true if and only if

$$\mathbb{E}[\log|\zeta(Z_{T_{1/2}})] = 0.$$

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References

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[5] Titchmarsh, E.C. *The Theory of the Riemann Zeta-Function* (revised by D.R. Heath-Brown). Clarendon, Oxford (1986).

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