

Ordinary abelian varieties over a finite field

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Translator's note.

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What follows is a translation of the French paper:

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We give here a down-to-earth description of the category of ordinary abelian varieties over a finite field \mathbb{F}_q . The result that we obtain was inspired by Ihara [2, ch. V] (see also [3]).

1

Let p be a prime number, \mathbb{F}_p the field $\mathbb{Z}/(p)$, and $\bar{\mathbb{F}}_p$ an algebraic closure of \mathbb{F}_p . For every power q of p , let \mathbb{F}_q be the subfield of q elements of $\bar{\mathbb{F}}_p$. For every algebraic extension k of \mathbb{F}_p , we denote by $W_0(k)$ the discrete valuation Henselian ring essentially of finite type over \mathbb{Z} , absolutely unramified, with residue field k ; let $W(k)$ be the ring of Witt vectors over k , i.e. the completion of $W_0(k)$. Let $W = W(\bar{\mathbb{F}}_p)$, and let φ be an embedding of W into the field \mathbb{C} of complex numbers. We denote by $\mathbb{Z}(1)$ the subgroup $2\pi i\mathbb{Z}$ of \mathbb{C} . The exponential map defines an isomorphism between $\mathbb{Z}(1) \otimes \mathbb{Z}_\ell$ and $\varprojlim \mu_{\ell^n}(\mathbb{C})$.

We denote by A^* the dual abelian variety of an abelian variety A . For every field k , we denote by \bar{k} the algebraic closure of k .

2

Let A be an abelian variety of dimension g , defined over a field k of characteristic p . Recall that A is said to be *ordinary* if any of the following equivalent conditions are satisfied:

- (I) A has p^g points of order dividing p with values in \bar{k} .
- (II) The "Hasse-Witt matrix" $F^* : H^1(A^{(p)}, \mathcal{O}_{A^{(p)}}) \rightarrow H^1(A, \mathcal{O}_A)$ is invertible.
- (III) The neutral component of the group scheme A_p that is the kernel of multiplication by p is of multiplicative type (and thus geometrically isomorphic to a power of μ_p).

If $k = \mathbb{F}_q$, and if F is the Frobenius endomorphism of A , and $\text{Pc}_A(F; x)$ is its characteristic polynomial, then these conditions are then equivalent to:

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- (IV) At least half of the roots of $\text{Pc}_A(F; X)$ in $\overline{\mathbb{Q}}_p$ are p -adic units. In other words, if $n = \dim A$, then the reduction $\pmod p$ of the polynomial $\text{Pc}_A(F; x)$ is not divisible by x^{n+1} .

3

Let A be an ordinary abelian variety over $\overline{\mathbb{F}}_p$. We denote by \tilde{A} the canonical Serre–Tate covering [4] of A over W . Recall that \tilde{A} depends functorially on A , and is characterised by the fact that the p -divisible group $T_p(\tilde{A})$ over W attached to \tilde{A} [5] is the product of the p -divisible groups (uniquely determined, by 2.(III)) that cover, respectively, the neutral component and the largest étale quotient of $T_p(A)$. The canonical covering \tilde{A} is again the unique covering of A such that every endomorphism of A lifts to \tilde{A} . We denote by $T(A)$ the integer homology of the complex abelian variety $A_{\mathbb{C}}$ induced by \tilde{A} and φ by the extension of scalars of W to \mathbb{C} : | p. 239

$$T(A) = H_1(\tilde{A} \otimes_{\varphi} \mathbb{C}).$$

We know that \tilde{A} descends uniquely to $W_0(\overline{\mathbb{F}}_p)$, and so $A_{\mathbb{C}}$ depends only on A and on the restriction of φ to $W_0(\overline{\mathbb{F}}_p)$. The free \mathbb{Z} -module $T(A)$ is of rank $2\dim(A)$; it is functorial in A . Furthermore, if $\ell \neq p$ is a prime number, then we have, functorially, that

$$T(A) \otimes \mathbb{Z}_{\ell} = T_{\ell}(A). \quad (3.1)$$

The canonical covering of the dual abelian variety A^* of A is the dual of \tilde{A} , and so $(A_{\mathbb{C}})^* = A_{\mathbb{C}}^*$, and $T(A)$ and $T(A^*)$ are in perfect duality with values in $\mathbb{Z}(1)$:

$$T(A) \otimes T(A^*) \rightarrow \mathbb{Z}(1) \quad (3.2)$$

(it is necessary to use $\mathbb{Z}(1)$ instead of \mathbb{Z} in order to obtain a theory that is invariant under complex conjugation). The pairings (3.2) are compatible, via (3.1), with the pairings

$$T_{\ell}(A) \otimes T_{\ell}(A^*) \rightarrow \mathbb{Z}_{\ell}(1);$$

a morphism $\xi: A \rightarrow A^*$ defines a polarisation of A if and only if $\xi_{\mathbb{C}}: A_{\mathbb{C}} \rightarrow A_{\mathbb{C}}^*$ defines a polarisation of $A_{\mathbb{C}}$. Set

$$\begin{aligned} T'_p(A) &= \text{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, A(\overline{\mathbb{F}}_p)) \\ T''_p(A) &= \text{Hom}_{\mathbb{Z}_p}(T'_p(A^*), \mathbb{Z}(1) \otimes \mathbb{Z}_p) \end{aligned}$$

These \mathbb{Z}_p -modules are covariant functors in A .

By definition of the canonical covering, the p -divisible group $T_p(\tilde{A})$ is the sum of the constant proétale group $T'_p(A)$ and the Cartier dual of $T'_p(A^*)$. For every morphism $u: A \rightarrow B$, the induced morphism $u: T_p(\tilde{A}) \rightarrow T_p(\tilde{B})$ can be identified with the sum of $u|T'_p(A): T'_p(A) \rightarrow T'_p(B)$ and the Cartier transpose of $u^t|T'_p(B^*): T'_p(B^*) \rightarrow T'_p(A^*)$. Over \mathbb{C} , we canonically have that $\mathbb{Z}(1)/(p^n) \sim \mu_{p^n}$, whence an isomorphism of functors:

$$T_{(p)}(A) = T(A) \otimes \mathbb{Z}_p = T'_p(A) \oplus T''_p(A). \quad (3.3)$$

4

Recall that, if $\varphi: X \rightarrow Y$ is an isogeny between complex abelian varieties, then the exact homotopy sequence reduces to a short exact sequence:

$$0 \rightarrow H_1(X) \rightarrow H_1(Y) \rightarrow \text{Ker}(\varphi) \rightarrow 0.$$

The abelian varieties that are quotients of X by a finite subgroup, and these finite subgroups of X , correspond bijectively with the sub-lattice of $H_1(X) \otimes \mathbb{Q}$ containing $H_1(X)$. | p. 240

Let A be an ordinary abelian variety over $\overline{\mathbb{F}}_p$. If n is an integer coprime to p , then the subschemes of finite groups of order n of A , of \tilde{A} , and of $A_{\mathbb{C}}$, correspond bijectively, and also correspond to lattices R containing $T(A)$ such that $[R : T(A)] = n$.

Set $V'_p = T'_p(A) \otimes \mathbb{Q}_p$ and $V''_p(A) = T''_p(A) \otimes \mathbb{Q}_p$. The subschemes of finite groups of order p^k of A are products of a étale subgroup and an infinitesimal subgroup. The étale subgroups of order p^k of A correspond to those of subgroups of order p^k of $A_{\mathbb{C}}$ such that the lattice R corresponding to $T(A)$ is contained inside $T_{(p)}(A) + V'_p(A)$. By duality, the infinitesimal subgroups of A correspond to the lattices R containing $T(A)$ that are p -isogenous to $T(A)$, i.e. such that $[R : T(A)]$ is a power of p and is contained in $T_{(p)}(A) + V''_p(A)$.

All told, the finite subgroups of A^p , or the abelian varieties that are quotients of A , correspond bijectively to the lattices R containing $T(A)$ such that

$$R \otimes \mathbb{Z}_p = (R \otimes \mathbb{Z}_p \cap V'_p) + (R \otimes \mathbb{Z}_p \cap V''_p). \quad (4.1)$$

5

In particular, $A^{(p)}$, the quotient of A by the largest infinitesimal subgroup of A that is annihilated by p (for ordinary A), is defined by the lattice $T(A)^{(p)}$ containing $T(A)$ that is p -isogenous to $T(A)$, and such that

$$T(A)^{(p)} \otimes \mathbb{Z}_p = T'_p(A) + \frac{1}{p}T''_p(A).$$

6

Let A be an abelian variety over \mathbb{F}_q , and $F: x \mapsto x^q$ its Frobenius endomorphism. Recall that A is uniquely determined by the pair (\overline{A}, F) induced by (A, F) by extension of scalars from \mathbb{F}_q to $\overline{\mathbb{F}}_q$; the endomorphism F of \overline{A} factors as the relative Frobenius morphism $F_r^{(q)}: \overline{A} \rightarrow \overline{A}^{(q)}$ followed by an isomorphism $F': \overline{A}^{(q)} \rightarrow \overline{A}$. If A is ordinary, then we denote by $T(A)$ the \mathbb{Z} -module $T(\overline{A})$ endowed with the endomorphism F induced by the Frobenius endomorphism of A . By §5, the above, and (3.3), the lattices $T(A)$ and $F(T(A))$ are p -isogenous, and we have that

$$F(T'_p(A)) = T'_p(A), \quad (6.1)$$

$$F(T''_p(A)) = qT''_p(A). \quad (6.2)$$

7

Theorem. *The functor $A \mapsto (T(A), F)$ is an equivalence of categories between the category of ordinary abelian varieties over \mathbb{F}_q and the category of free \mathbb{Z} -modules T of finite type endowed with an endomorphism F that satisfy the following conditions:*

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- (a) F is semi-simple, and its eigenvalues have complex absolute value $q^{\frac{1}{2}}$,
- (b) at least half of the roots in $\overline{\mathbb{Q}}_p$ of the characteristic polynomial of F are p -adic units; in other words, if T is of rank d , then the reduction \pmod{p} of the polynomial $\text{Pc}_T(F; x)$ is not divisible by $x^{\lfloor d/2 \rfloor + 1}$,
- (c) there exists an endomorphism V of T such that $FV = q$.

If condition (a) is satisfied, then conditions (b) and (c) are equivalent to the following:

- (d) the module $T \otimes \mathbb{Z}_p$ admits a decomposition, stable under F , into two sub- \mathbb{Z}_p -modules T'_p and T''_p of equal dimension, and such that $F|_{T'_p}$ is invertible, and $F|_{T''_p}$ is divisible by q .

Proof. (A) We first prove that (a)+(b)+(c) \implies (d). If α is a complex eigenvalue of F , then $\bar{\alpha}$ is another, of the same multiplicity, and $\alpha\bar{\alpha} = q$. If we exclude those that are equal to $\pm q^{\frac{1}{2}}$, then the eigenvalues of F in \mathbb{C} , and thus in $\overline{\mathbb{Q}}_p$, can be grouped into pairs of roots α and q/α . The roots α and q/α can not simultaneously be p -adic units, and so it follows from (b) that $\pm q^{\frac{1}{2}}$ is not an eigenvalue of F , that half of the eigenvalues of F in $\overline{\mathbb{Q}}_p$ are p -adic units, say $\alpha_1, \dots, \alpha_{d/2}$, and that the other half are of the form $\beta_1 = q/\alpha_1, \dots, \beta_{d/2} = q/\alpha_{d/2}$. Let $T_{(p)} = T \otimes \mathbb{Z}_p$, $V_p = T \otimes \mathbb{Q}_p$, V'_p the subspace of V_p given by the kernel of $\prod_i (F - \alpha_i)$, and V''_p the kernel of the endomorphism $\varphi = \prod_i (F - \beta_i)$. We have that $V_p = V'_p \oplus V''_p$. Let T'_p be the projection from $T_{(p)}$ to V'_p , and let $T''_p = T_{(p)} \cap V''_p$. Since φ annihilates V''_p , and respects T , it sends T'_p to $T_{(p)} \cap V'_p \subset T'_p$. Also, $\det(\varphi|_{V''_p}) = \prod_{i,j} (\alpha_i - \beta_j)$ is a p -adic unit, and so $\varphi(T'_p) = T'_p$, and $T_{(p)} \cap V'_p = T'_p$, and so $T_{(p)} = T'_p \oplus T''_p$.

(B) *Full faithfulness.* Let A and B be abelian varieties over \mathbb{F}_q , and let ψ be the arrow

$$\psi: \text{Hom}(A, B) \rightarrow \text{Hom}_F(T(A), T(B)).$$

By the theorem of Tate [7] and by (3.1), the arrow

$$\psi_\ell: \text{Hom}(A, B) \otimes \mathbb{Z}_\ell \rightarrow \text{Hom}_F(T(A), T(B)) \otimes \mathbb{Z}_\ell$$

is an isomorphism for $(\ell, p) = 1$, and so $\psi \otimes \mathbb{Q}$ is an isomorphism. We know that $\text{Hom}(A, B)$ is torsion free, and so ψ is injective. Let $u: A \rightarrow B$ be a morphism such that $T(u)$ is divisible by n . The induced morphism $u_\mathbb{C}: \overline{A}_\mathbb{C} \rightarrow \overline{B}_\mathbb{C}$ is thus divisible by n , and thus so too is $\tilde{u}: \tilde{A} \rightarrow \tilde{B}$ at the generic point of W . The kernel of multiplication by n is flat over W ; \tilde{u} thus disappears on this kernel, \tilde{u} and u are divisible by n , and ψ is bijective.

(C) *Necessity.* The fact that $(T(A), F)$ satisfies (a) follows from Weil; condition (d), which implies (b) and (c), follows from §6.

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- (D) *Isogenies.* Let (T_0, F) satisfy (a) and (d), and let T be a lattice in $T_0 \otimes \mathbb{Q}$, stable under F , that also satisfies (d). Suppose that (T_0, F) is the image of an abelian variety A over \mathbb{F}_q ; we will prove that (T, F) comes from an isogenous abelian variety. By T with $\frac{1}{k}T$, which is isomorphic to T , we can suppose that $T \supset T_0$. Condition (d) implies that T satisfies (4.1), and that T defines a subgroup H of \overline{A} , defined over \mathbb{F}_q , and such that $(T, F) = T(A/H)$.
- (E) *Surjectivity.* The functor T induces a functor $T_{\mathbb{Q}}$ from the category of isogeny classes of ordinary abelian varieties over \mathbb{F}_q to the category of finite-dimensional \mathbb{Q} -vector spaces endowed with an automorphism F that satisfies (a) and (b). By (D), it suffices to prove that this functor $T_{\mathbb{Q}}$ is essentially surjective. It even suffices to show that every simple object (V, F) in the codomain is in the image. By Honda [1] (see also [6]), there exists an abelian variety A over \mathbb{F}_q such that the characteristic polynomial of the Frobenius F_A of A is a power of that of F . The third characterisation in §2 of ordinary abelian varieties shows that A is ordinary. Furthermore, $(T(A) \otimes \mathbb{Q}, F)$ is the sum of copies of (V, F) , and thus, by (B), the isogeny class of the abelian variety $A \otimes \mathbb{Q}$ is the sum of copies of an abelian variety B that satisfies $T(B) \otimes \mathbb{Q} = (V, F)$. \square

8

Let (T, F) be a pair satisfying the hypotheses of the theorem, $2g$ the rank of T , A the corresponding abelian variety over \mathbb{F}_q , and $A_{\mathbb{C}}$ the induced complex abelian variety (§3). We have that

$$T = H_1(A_{\mathbb{C}}),$$

and so $T \otimes \mathbb{R}$ can be identified with the Lie algebra of $A_{\mathbb{C}}$, and is thus endowed with a complex structure. Here, thanks to J.-P. Serre, is how to reconstruct this complex structure in terms of T , F , and the restriction of φ to $W_0(\mathbb{F}_p)$:

Proposition. *The complex structure on $T \otimes \mathbb{R}$ defined above is characterised by the following properties:*

- (I) *The endomorphism F is \mathbb{C} -linear.*
- (II) *If v is the valuation of the algebraic closure $\overline{\mathbb{Q}}$ of \mathbb{Q} in \mathbb{C} that extends the valuation of $W_0(\mathbb{F}_p)$, then the valuations of the g eigenvalues of this endomorphism are strictly positive.*

Proof. Condition (I) is evident, and condition (II) follows from the fact that the action of F on the Lie algebra of A is congruent to zero mod p . The uniqueness of a structure satisfying (I) and (II) follows easily from condition (b), satisfied by (T, F) . \square

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References

- [1] Honda, T. Isogeny classes of abelian varieties over finite fields. *J. Math. Soc. Jap.* **20** (1968), 83–95.

- [2] Ihara, Y. *On congruence monodromy problems, vol. I*. University of Tokyo, 1968.
- [3] Ihara, Y. The congruence monodromy problems. *J. Math. Soc. Jap.* **20** (1968), 107–121.
- [4] Lubin, J., Serre, J.-P., and Tate, J. “Elliptic curves and formal groups”. Woods Hole Summer Institute 1964 (mimeographed, printed in a limited number of copies).
- [5] Serre, J.P. “Groups p -divisibles (d’après J. Tate)”. *Séminaire Bourbaki* **10** (1966–67), Talk no. 318.
- [6] Tate, J. “Classes d’isogénies de variétés abéliennes sur un corps fini (d’après T. Honda)”. *Séminaire Bourbaki* **11** (1968–69), Talk no. 352.
- [7] Tate, J. Endomorphisms of abelian varieties over finite fields. *Inventiones Math.* **2** (1966), 134–144.