The fundamental group of the projective line minus three points

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Translator's note

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Introduction

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The present article owes much to A. Grothendieck. He invented the philosophy of motives, which is our guiding thread. Around five years ago, he also said to me, with conviction, that the profinite completion $\hat{\pi}_1$ of the fundamental group of $X := \mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}$, with the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, is a remarkable object, and that it must be studied.

Every finite cover of X can be described by equations with coefficients in the algebraic numbers. Applying an element of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to these coefficients, we obtain the equations of another cover. Understanding how $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ permutes the isomorphism classes of finite covers essentially reduces to understanding the Galois action on $\hat{\pi}_1$. "Essentially," since I have omitted mentioning the base points, and since the Galois covers have not been thought of as G-covers, for G their automorphism group.

Up until now, we have not had the language necessary to study the Galois action on $\hat{\pi}_1$. A. Grothendieck and his students have developed a combinatorial description ("charts") of finite covers of X, based on a decomposition of $\mathbb{P}^1(\mathbb{C})$ into the two "spherical triangles'' $\Im(z) \geqslant 0$ and $\Im(z) \leqslant 0$, with sides $[\infty, 0]$, $[0,1]$, and $[1,\infty]$. This has not helped in understanding the Galois action. We have only a few unresolved examples of covers whose Galois conjugates have been calculated.

In this article, we only consider when $\hat{\pi}_1$ is rendered nilpotent, i.e. quotients $\hat{\pi}_1^{(N)}$ of $\hat{\pi}_1$ by the subgroups of its decreasing central series. The profinite group $\hat{\pi}_1^{(N)}$ is a product over primes ℓ of nilpotent pro-ℓ-groups: $\hat{\pi}_1^{(N)} = \prod_\ell \hat{\pi}_1^{(N)}{}_\ell$. Each $\hat{\pi}_1^{(N)}{}_\ell$ is an ℓ-adic Lie group. It admits a Lie algebra $\operatorname{Lie} \hat{\pi}_1^{(N)}{}_\ell$, which is a Lie algebra over $\mathbb{Q}_\ell$. If we choose a base point $x \in X(\mathbb{Q}) = \mathbb{Q} \setminus \{0,1\}$, then $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on these Lie algebras. The action, up to inner automorphism, does not depend on the choice of x. We would like to understand these actions.

The nilpotent versions of π_1 are very close to cohomology. This is most visible in the theory of D. Sullivan [**Su?**,**Mo?**]. *Notation:* for Γ a finitely generated group, let $Z^i\Gamma$ be the decreasing central series, let $\Gamma^{(N)} = \Gamma/Z^{N+1}\Gamma$, and let $\Gamma^{[N]} = \Gamma^{(N)}/\text{torsion}$ (9.3). The theory of Malcev [**Mal?**] attaches a nilpotent Lie algebra over $\mathbb{Q}$, denoted $\operatorname{Lie}\Gamma^{[N]}$, to $\Gamma^{[N]}$, such that $\Gamma^{[N]}$ is a congruence subgroup of the unipotent algebraic group over $\mathbb{Q}$ of the Lie algebra $\operatorname{Lie}\Gamma^{[N]}$. By D. Sullivan, if X is a differentiable manifold, then $\operatorname{Lie}\pi_1(X)^{[N]} \otimes \mathbb{R}$ is determined, up to inner automorphism, by the differential-graded algebra $\Omega^\bullet_X$, taken up to quasi-isomorphism. *p. 4 (82)*

This close relation with cohomology hints that the study of nilpotent versions of $\hat{\pi}_1$ is far from the "anabelian" dream of A. Grothendieck. It allows us, however, to use his philosophy of motives.

Let k be a number field. If X is an algebraic variety over k, then we have a whole series of parallel cohomology theories for X: the classical cohomology of $X(\mathbb{C})$ (for each complex embedding of k), crystalline cohomology (which is equal to de Rham cohomology if X is smooth), ℓ-adic cohomology, … The groups thus obtained are endowed with various additional structures (Hodge mixed, Galois action, …) and are linked by comparison isomorphism. In §1, we axiomatise the situation by defining "realisation systems over k." The exact definition is not to be taken seriously: considering the applications — and what we are capable of doing — it could be wise to either add or remove data as much as axioms. The essential, for us, is that

i. The category of realisation systems is endowed with a $\otimes$ satisfying the usual properties: it is a Tannakian category over $\mathbb{Q}$.

ii. Conjecturally, the category of motives is a full subcategory of the category of realisation systems.

Condition (ii) requires, in particular, that, for every variety X over k and for every i, the available cohomology theories, applied to X, give a realisation system $H^i(X)$ over k (which we will denote by $H^i(X)_{\text{mot}}$, and call the motivic H^i of X).

Analogous ideas have been independently developed by U. Jannsen [**J?**]. In [**J?**], U. Jannsen defines (mixed) motives over k as constituting the Tannakian subcategory (of the category of realisation systems) generated by the $H^i(X)$ for X smooth and quasi-projective. Here we are still being imprecise, saying that a motive over k is a realisation system "of geometric origin." For X over k and $x \in X(k)$, we want, for example, to regard $\mathrm{Lie}\,\pi_1(X(\mathbb{C}),x)^{[N]}$ as a realisation of a motive over k.

This article owes much to an unpublished work of Z. Wojtkoviak. For $X = \mathbb{P}^1 \setminus \{0,1,\infty\}$ and $x \in X(\mathbb{C})$, I proposed to him a definition of the mixed Hodge structure of $\mathrm{Lie}\,\pi_1(X(\mathbb{C}),x)^{[N]}$. He calculated it in part, for small N, and, to my extreme surprise, show that, for $N = 4$, its description involves $\zeta(3)$. A decanted form of the calculations appear in §19. In fact, the whole article originates from my desire to understand the result of Z. Wojtkoviak. I have also been helped by the answer by O. Gabber to my question "How can we construct an extension of $\mathbb{Z}_\ell$ by $\mathbb{Z}_\ell(3)$, uniformly in ℓ?": "By a class in $K_5(\mathbb{Q})$," as well as by the conjectures of A. Beilinson on the values of L-functions. *p. 5 (83)*

If X is an algebraic variety over a number field k, $x \in X(k)$, and N an integer, then we want to have a realisation system $\mathrm{Lie}\,\pi_1(X,x)^{(N)}_{\mathrm{mot}}$. We can only succeed in constructing this under additional hypotheses on X: in the general case, certain realisations are missing. The case of $\mathbb{P}^1$ minus some points — more generally, of smooth rational varieties — is nonetheless covered.

Let $k = \mathbb{Q}$, $X = \mathbb{P}^1 \setminus \{0,1,\infty\}$, and $x \in X(\mathbb{Q})$. The associated graded algebra for the weight filtration of $\mathrm{Lie}\,\pi_1(X,x)^{(N)}_{\mathrm{mot}}$ is the free Lie algebra on $H_1(X)_{\mathrm{mot}}$, modulo its Z^{N+1} (decreasing central series). $H_1(X)_{\mathrm{mot}}$ is the sum of two copies of the Tate motive $\mathbb{Q}(1)$. We thus deduce that $\mathrm{Lie}\,\pi_1(X,x)^{(N)}_{\mathrm{mot}}$ is an iterated extension of Tate motives $\mathbb{Q}(n)$. The fact that non-trivial extensions appear is what gives it its charm.

I conjecture that, over a number field k, the group of motivic extensions of $\mathbb{Q}$ by $\mathbb{Q}(n)$ ($n > 0$) is $K_{2n-1}(k) \otimes \mathbb{Q}$. For a general framework into which we can place this conjecture, see [**B?**]. In particular, for $k = \mathbb{Q}$, we want $\mathrm{Ext}^1(\mathbb{Q},\mathbb{Q}(n))$ to be zero for n even, and of dimension 1 for $n \geqslant 3$ odd. This is the motivic Ext^1: extensions as realisation systems that "come from algebraic geometry." This conjecture places severe restrictions on $\mathrm{Lie}\,\pi_1(X,x)^{(N)}_{\mathrm{mot}}$, which are far from having been verified. What we know concerns, up to now, only the quotient by the second derived group. A large part of this article is dedicated to developing a language in which the consequences of the conjecture affecting $\mathrm{Lie}\,\pi_1(X,x)^{(N)}_{\mathrm{mot}}$ can be clearly stated.

We now go through this article, pointing out several shortcuts.

In §1, we describe the category of realisation systems over a base S. The base S can be: $\mathrm{Spec}(\mathbb{Q})$, $\mathrm{Spec}(\mathbb{F})$ for $\mathbb{F}$ a number field, an open subset of the spectrum of the ring of integers of a number field, or smooth over $\mathrm{Spec}(\mathbb{Z})$. In this category, the Hom are $\mathbb{Q}$-vector spaces. We also define a notion of integer structure; in the category of realisation systems with integer coefficients (= endowed with an integer structure), the Hom are free $\mathbb{Z}$-modules of finite type. The definition has a crystalline component. The reader is invited to ignore this for a first approximation. The theory coincides with that of U. Jannsen [**J?**]. The crystalline aspect will be neglected in the rest of the introduction. *p. 6 (84)*

In §2 we give examples. We also explain what an extension of the unit realisation system $\mathbb{Z}$ by a realisation system M with integer coefficients is. *Terminology: M-torsor*, or torsor under M. *Example:* the Kummer $\mathbb{Z}(1)$-torsor, where $\mathbb{Z}(1)$ is the Tate motive.

In §3 we describe certain remarkable torsors, which can be said to be cyclotomic, under the Tate motive $\mathbb{Z}(k)$. §16 explains how these torsors naturally appear in the study of π_1 of $\mathbb{P}^1 \setminus \{0,1,\infty\}$. The description here is direct, but unmotivated. The claim that some of these torsors are of finite order ((3.5), (3.14)) lets us recover the known formulas expressing the Dirichlet L-functions in negative integers as integrals of distributions over $\widehat{\mathbb{Z}}$ with values in $\widehat{\mathbb{Z}}$: a version of Kummer congruences. In §18, we prove (3.5) and (3.14) using the geometric interpretation of §16. In §3, we give a direct proof, by using the known formulas for $L(\chi, 1-k)$.

§4 is a pot-pourri of reminders on Ind-objects and pro-objects. The reader is invited to consult this only when needed.

We want to give a motivic sense to an assertion like the following: the fundamental group of $\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}$ (at base point b) is freely generated by the following loops:

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The purpose of §5, §7, and §15 is to construct the language which allows us to do this. This consists of

a. giving a motivic sense to $\pi_1(X,x)^{(N)}$, not only to its Lie algebra;

b. giving a motivic sense to the torsor (0.6) of homotopy classes of paths from b_1 to b_2;

c. in (1), the "monodromy around 0" loop is only unambiguously determined for b "close to 0." We must define what it means for a base point to be "close to 0."

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Our solution will be to define a motivic linear group as being an Ind-object in the category of motives, endowed with the structure of a commutative Hopf algebra. To avoid speculation: consider the group in realisation systems, and replace "motive" by "realisation system." There is an analogous definition for torsors under a group. We separately define a notion of "integer" structures. This definition has the advantage that the standard constructions in algebraic geometry (decreasing central series, quotients, pushing forward a G-torsor by $G \to H$, twisting by a torsor, ...) all translate automatically to the motivic case. This, in an arbitrary Tannakian category, is explained in §5.

In §7, we reinterpret these definitions in a language that is closer to that of our applications. The reader who is displeased by the general nonsense of §5 and §7 can take the interpretations given in §7 as the definition of groups, torsors, ... in realisation systems. Drawback: every standard construction must be redefined in this case.

In the classical definition of π_1, the role of the base point b can be played by a contractible subset B. It can also be played by a filter $\mathscr{B}$ on X whose base if given by contractible subsets. For example, if X is a Riemann surface $\overline{X}$ minus a point s, and v is a non-zero tangent vector at s, with z being a local coordinates centred at s, then we can take the contractible subsets

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The filter $\mathscr{B}(v)$ that they generate is independent of the chosen coordinate. By this construction, a non-zero tangent vector at s can act as a base point in the definition of π_1 of X.

The same phenomenon occurs in the profinite theory of π_1, and in the "de Rham" theory. Be aware that $\mathscr{B}(v) = \mathscr{B}(\lambda v)$ for real $\lambda > 0$, but that this fact has no analogue in the other theories. There constructions are explained in §15. They allow us, in the definition of the motivic π_1 of X, to take a base point "at infinity," like the tangent vector v at s. p. 8 (86)

Let $X = \mathbb{P}^1 \setminus \{0,1,\infty\}$. An algebraic meaning of "base point close to 0" is "non-zero tangent vector at 0." For such a base point b, the monodromy around 0 has a motivic meaning: it is a morphism of motivic groups

$$\mathbb{Z}(1) \to \pi_1(X,b)_{\mathrm{mot}}.$$

Here and later on, π_1 is the pro-unipotent π_1, defined as the projective limit of the motivic groups $\pi_1(X,b)^{(N)}_{\mathrm{mot}}$.

We take the base point to be the tangent vector 1 at 0. We have a good reduction mod p for every p, and $\pi_1(X,b)^{(N)}_{\mathrm{mot}}$ is a linear group in the Tannakian category of motives over Spec($\mathbb{Z}$) that are iterated extensions of Tate motives. §8 states a conjecture on the $\mathrm{Ext}^1(\mathbb{Q},\mathbb{Q}(k))$ in this category, as well as some consequences. At the end of §16, we make these explicit in the case of $\pi_1(X,b)^{(N)}_{\mathrm{mot}}$. I hope that this places the $\zeta(3)$ discovered by Z. Wojtkoviak in its natural setting. §6 is preliminary. For the essential idea, see (6.2).

To define the motivic π_1, we need to patch together the various theories of π_1 that we have at our disposal, guided by the goal of constructing a motivic group in the sense of §5, explained in §7. This is done in §10 to §13, after a reminder (§9) on the Malčev theory of nilpotent groups and their Lie algebras. The result leaves much to be desired. It is only completely studied for smooth algebraic varieties whose smooth compactifications $\overline{X}$ satisfy $H^1(\overline{X},\mathscr{O}) = 0$. Another complaint: I sometimes only sketch the definition of structures that will be used in future calculations.

In §16, we finally explain what the $\mathbb{Z}(k)$-torsors from §3 have to do with the π_1 of the projective line minus three points. The justifying calculations are given in §19. We give, in §17 and §18, a geometric explanation of some of their properties.

Terminology and notation

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0.1. We denote inductive limits and projective limits by lim ind and lim proj.

0.2. For a prime number ℓ, we denote by $\mathbb{Z}_\ell$ and $\mathbb{Q}_\ell$ the completions of $\mathbb{Z}$ and $\mathbb{Q}$ for the ℓ-adic topology:

$$\mathbb{Z}_\ell = \lim\operatorname{proj} \mathbb{Z}/\ell^n\mathbb{Z},$$
$$\mathbb{Q}_\ell = \mathbb{Z}_\ell \otimes \mathbb{Q}.$$

We denote by $\widehat{\mathbb{Z}}$ the profinite completion of $\mathbb{Z}$, and by $\mathbb{A}^{\mathrm{f}}$ the ring of finite adeles:

$$\widehat{\mathbb{Z}} \xrightarrow{\sim} \prod_\ell \mathbb{Z}_\ell,$$
$$\mathbb{A}^{\mathrm{f}} = \widehat{\mathbb{Z}} \otimes \mathbb{Q}.$$

We denote by $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

0.3. For an abstract group, algebraic group, profinite group, or Lie algebra A, we denote by $Z^i(A)$ the descending central series. We use the numbering for which $A = Z^1(A)$. We denote by $A^{(N)}$ the quotient of A by $Z^{N+1}(A)$. In the case of abstract or profinite groups, we denote by $A^{[N]}$ the largest torsion-free quotient of $A^{(N)}$.

0.4. We denote by $\otimes$ an extension of scalars. For example, if X is a scheme over k, and k' is an extension of k, then we set

$$X \otimes k' := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k').$$

0.5. Let G be a sheaf of groups on a site $\mathscr{S}$, or, equivalently, in a topos T. Useful particular case: if $\mathscr{S}$ is a point, then a sheaf is a set and G is a group. A *G-torsor*, or *torsor under G*, is a sheaf P endowed with a right G-action such that P is locally isomorphic to G acting on itself by translations on the right. We also call such an object a *right G-principal homogeneous space*, or a *right principal homogeneous space under G*. If P is a G-torsor, then a sheaf X on which G acts can be *twisted* by P. The twisting X^P is the contracted product $P \times^G X = (P \times X)/G$, and is endowed with $\alpha\colon P \to \underline{\mathrm{Isom}}(X, X^P)$ satisfying $\alpha(pg) = \alpha(p)g$. *p. 10 (88)*

An *(H,G)-bitorsor* (cf. SGA 7, VII.1, or Girard, *Cohomologie non abelienne*, III 1.5) is a space which is simultaneously a left principal homogeneous space under H and a right principal homogeneous space under G, with the G- and H-actions commuting with one another. If P is a G-torsor, then the sheaf of automorphisms of P is the twisting G^P of G by P (under the action of G on itself by inner automorphisms), and P is a (G^P, P)-bitorsor. By this construction, the data of an (H,G)-bitorsor P is equivalent to the data of a G-torsor P along with an isomorphism between H and G^P. *Notation:* we will write ${}_HP_G$ to mean that P is an (H,G)-bitorsor.

We will use the following operations on torsors and bitorsors.

- **Pushing forward:** (or **transporting**) a G-torsor P by $\varphi\colon G \to H$ to obtain an H-torsor $\varphi(P)$. A *φ-morphism* from the G-torsor P to the H-torsor Q is some $u\colon P \to Q$ such that $u(pg) = u(p)\varphi(g)$. A φ-morphism factors uniquely through an isomorphism of H-torsors between $\varphi(P)$ and Q.
- **Composition:** of a (G_1, G_2)-bitorsor P and a (G_2, G_3)-bitorsor Q: the (G_1, G_3)-bitorsor $P \circ Q$ given by the contracted product $P \times^{G_2} Q = (P \times Q)/G_2$.
- **Inverse:** of ${}_{G_1}P_{G_2}$: the (G_2, G_1)-bitorsor P^{-1}, unique up to isomorphism, endowed with $(p \mapsto p^{-1})\colon P \to P^{-1}$ such that $(g_1 p g_2)^{-1} = g_2^{-1} p^{-1} g_1^{-1}$.

  For G-torsors P and Q, the sheaf $\underline{\mathrm{Isom}}(P,Q)$ of isomorphisms of G-torsors from P to Q is the (G^Q, G^P)-bitorsor $G \circ P^{-1}$.

If the site $\mathscr{S}$ is such that the representable functors h_S are sheaves, then we can transport these operations to $\mathscr{S}$ via the fully faithful functor $S \mapsto h_S$, with each construction only being defined if it does not leave the collection of representable sheaves.

1 Mixed Motives

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1.1. For algebraic varieties, we have various parallel cohomology theories. The most important for us will be de Rham and ℓ-adic cohomology.

- **De Rham cohomology.** Let k be a field of characteristic 0, and X an algebraic variety over k. Suppose that X is smooth. The de Rham cohomology groups $\mathrm{H}^i_{\mathrm{DR}}(X)$ are the hypercohomology groups of the de Rham complex:

  $$\mathrm{H}^i_{\mathrm{DR}}(X) := \mathbb{H}^i(X, \Omega^\bullet_{X/k})$$

  cf. [**G?**]. These are vector spaces over k. If k' is an extension of k, and X' over k' is given by extension of scalars of X, then

  $$\mathrm{H}^i_{\mathrm{DR}}(X') = \mathrm{H}^i_{\mathrm{DR}}(X) \otimes_k k'.$$

  If X is not smooth, then the de Rham complex no longer gives a reasonable theory. We can define the $\mathrm{H}^i_{\mathrm{DR}}(X)$ by reduction to the smooth case, by the methods of [**D3?**], or, if X admits an embedding into a smooth variety Z, as the hypercohomology of the de Rham complex of the formal completion of Z along X (R. Hartshorne, *On the de Rham cohomology of algebraic varieties*, Publ. Math. IHES **45** (1975), p. 5–99); more intrinsically, it is the crystalline cohomology of X (A. Grothendieck, *Crystals and the de Rham cohomology of schemes*, Notes by J. Coates and O. Jussila, in: "dix exposés sur la cohomologie des schémas," North Holland (1968)).

- **ℓ-adic cohomology.** Let ℓ be a prime number; if k is an algebraically closed field of characteristic $\neq \ell$, then we have the ℓ-adic theory $X \mapsto \mathrm{H}^i(X, \mathbb{Q}_\ell)$ that associates, to X over k, cohomology groups which are vector spaces over $\mathbb{Q}_\ell$ (cf. SGA 5, VI). They are defined from the cohomology groups with coefficients in $\mathbb{Z}/(\ell^n)$, and we allow ourselves to give, as reference for a theorem in ℓ-adic cohomology, the place where its $\mathbb{Z}/(\ell^n)$ analogue is proved. The $\mathrm{H}^i(X, \mathbb{Q}_\ell)$ depend only on X. In particular, if k is the algebraic closure of k_0, and if X is given by extension of scalars of some X_0 over k_0, then $\mathrm{Gal}(k/k_0)$ acts (semi-k-linearly) on X, and thus acts on the $\mathrm{H}^i(X, \mathbb{Q}_\ell)$. This action is continuous. If k' is an algebraically closed extension of k, and if X' is given by extension of scalars of X, then $\mathrm{H}^i(X, \mathbb{Q}_\ell) \xrightarrow{\sim} \mathrm{H}^i(X', \mathbb{Q}_\ell)$. This follows by passing to p. 12 (90) the limit in the base change theorem for a smooth morphism [SGA 4, XVI, 1.2]: k' is the filtrant inductive limit of the k-algebras A with $\mathrm{Spec}(A)$ smooth over k.

If $k = \mathbb{C}$, then we have the topological space $X(\mathbb{C})$ of points of X, as well as its rational cohomology $\mathrm{H}^\bullet(X(\mathbb{C}), \mathbb{Q})$. We have canonical isomorphisms from [**G?**] and [SGA4, XVI, 4.1]:

$$\mathrm{H}^i_{\mathrm{DR}}(X) = \mathrm{H}^i(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \tag{1.1.1}$$

$$\mathrm{H}^i(X, \mathbb{Q}_\ell) = \mathrm{H}^i(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell. \tag{1.1.2}$$

If k is a field of characteristic 0, and $\sigma\colon k \to \mathbb{C}$ a complex embedding, with $\overline{k}$ the algebraic closure of k in $\mathbb{C}$ via σ, then we obtain the isomorphisms

$$\mathrm{H}^i_{\mathrm{DR}}(X) \otimes_{k,\sigma} \mathbb{C} = \mathrm{H}^i(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \tag{1.1.3}$$

$$\mathrm{H}^i(X \otimes \overline{k}, \mathbb{Q}_\ell) = \mathrm{H}^i(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \tag{1.1.4}$$

where $X(\mathbb{C})$ is the topological space of points of the complex algebraic variety given by the extension of scalars via σ of X.

The existence of parallel cohomology theories lead A. Grothendieck to conjecture the existence, for all base fields k, of a motivic theory $X \mapsto \mathrm{H}^i_{\mathrm{mot}}(X)$, defined on algebraic varieties (i.e. schemes of finite type) over k and with values in a category $\mathscr{M}(k)$ (to be defined) of motives over k. The known theories would then be deduced from the motivic theory by applying *realisation* functors.

The category $\mathscr{M}(k)$ should be an abelian category, with Hom groups of finite dimension over $\mathbb{Q}$. It should be endowed with a tensor product $\otimes\colon \mathscr{M}(k) \times \mathscr{M}(k) \to \mathscr{M}(k)$ and associativity and commutative data $(X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ and $X \otimes Y \to Y \otimes X$ satisfying the usual properties — more precisely, making $\mathscr{M}(k)$ into a Tannakian category [**S?**,**DM?**,**D4?**]. By the theory of Tannakian categories, $\mathscr{M}(k)$ would be the category of representations of a gerbe whose band is affine over Spec($\mathbb{Q}$). For k of characteristic 0, the category $\mathscr{M}(k)$ with its tensor product should be equivalent to the category of representations of an scheme in affine groups (i.e. a proalgebraically affine group) over $\mathbb{Q}$.

Each $\mathrm{H}^i_{\mathrm{mot}}(X)$ would be a contravariant functor in X. We should also have Künneth isomorphisms

$$\mathrm{H}^i_{\mathrm{mot}}(X \times Y) \simeq \bigoplus_{i=j+k} \mathrm{H}^j_{\mathrm{mot}}(X) \otimes \mathrm{H}^k_{\mathrm{mot}}(Y) \tag{1.1.5}$$

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giving rise to commutative diagrams

$$\begin{array}{ccc}
\mathrm{H}^i_{\mathrm{mot}}(X \times Y) & \xrightarrow{\longleftarrow} & \mathrm{H}^i_{\mathrm{mot}}(Y \times X) \\
\downarrow & & \downarrow \\
\mathrm{H}^j_{\mathrm{mot}}(X) \otimes \mathrm{H}^k_{\mathrm{mot}}(Y) & \xrightarrow[(-i)^{jk}]{} & \mathrm{H}^k_{\mathrm{mot}}(Y) \otimes \mathrm{H}^j_{\mathrm{mot}}(X)
\end{array}$$

$$\begin{array}{ccc}
\mathrm{H}^i_{\mathrm{mot}}((X \times Y) \times Z) & \xrightarrow{\longleftarrow} & \mathrm{H}^i_{\mathrm{mot}}(X \times (Y \times X)) \\
\downarrow & & \downarrow \\
(\mathrm{H}^j_{\mathrm{mot}}(X) \otimes \mathrm{H}^k_{\mathrm{mot}}(Y)) \otimes \mathrm{H}^\ell_{\mathrm{mot}}(Z) & \xrightarrow[\longleftarrow]{} & \mathrm{H}^j_{\mathrm{mot}}(X) \otimes (\mathrm{H}^k_{\mathrm{mot}}(Y) \otimes \mathrm{H}^\ell_{\mathrm{mot}}(Z)).
\end{array}$$

Each of the known cohomological theories should give rise to a "realisation" functor, compatible with the tensor product. For example, for k of characteristic 0, we would have

$$\mathrm{real}_{\mathrm{DR}}\colon \mathscr{M}(k) \to \text{vector spaces over } k$$

and, for X an algebraic variety over k, a functorial isomorphism

$$\mathrm{H}^i_{\mathrm{DR}}(X) = \mathrm{real}_{\mathrm{DR}}\, \mathrm{H}^i_{\mathrm{mot}}(X)$$

compatible with the Künneth isomorphisms.

The subcategory of $\mathscr{M}(k)$ *generated* by a set $\mathscr{M}$ of motives is defined to be the smallest full subcategory of $\mathscr{M}(k)$ containing $\mathscr{M}$ that is stable under $\oplus$, $\otimes$, taking the dual, and subquotients. If we only consider certain algebraic varieties X over k, then it can be useful to consider, instead of $\mathscr{M}(k)$, the subcategory generated by the $\mathrm{H}^i(X)$.

1.2. If we only consider smooth and projective varieties over a field k, and we assume the "standard" conjectures on algebraic cycles, then Grothendieck has shown how to define the category of motives generated by the $\mathrm{H}^i_{\mathrm{mot}}(X)$ (cf. [**Kl?**,**Man?**]); it is a semi-simple abelian category.

If we do not restrict ourselves to the category generated by the $\mathrm{H}^i_{\mathrm{mot}}(X)$ for X smooth and projective over k, then we no longer have even a conjectural definition of what the category of motives over k should be. However, the philosophy of motives is not made any less useful by this fact: it organises known facts, poses questions, and suggests precise conjectures.

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1.3. In each of the known theories, the $\mathrm{H}^i(X)$ are endowed with an increasing filtration W, known as the *weight filtration* [**D5?**], as well as comparison isomorphisms such that (1.1.1) and (1.1.2) are compatible with W. Furthermore, every natural map is strictly compatible with W. From this, we get a new requirement for the category of motives: every motive is endowed with a weight filtration W, compatible with the tensor product, and strictly compatible with every morphism $f : M \to N$, i.e.

$$f(M) \cap W_i(N) = f(W_i(M)).$$

We say that a motive M is *pure of weight* i if $W_i(M) = M$ and $W_{i-1}(M) = 0$. For X smooth and projective, $\mathrm{H}^i_{\mathrm{mot}}(X)$ is pure of weight i. We want for the $\otimes$-category generated by the $\mathrm{H}^i_{\mathrm{mot}}(X)$, for X smooth and projective over k, to be the sum of pure motives. In terms of pure motives, the properties of W can be written as follows: every motive is the iterated extension of pure motives, and, for M and N pure of weights m and n (respectively),

a. $M \otimes N$ is pure of weight $m+n$;
b. for $m \neq n$, $\mathrm{Hom}(M,N) = 0$; and
c. for $m \leqslant n$, $\mathrm{Ext}^1(M,N) = 0$.

Often, pure motives (or direct sums of pure motives) are simply called *motives*, and their category admits the conjectural description [**Kl?**,**Man?**]; the more general motives, considered here, are then called *mixed motives*

1.4.

If we cannot define the category of motives, we can at least describe a sequence of compatibilities between the $\mathrm{H}^i(X)$ taken in the various cohomological theories, i.e. describe compatibilities that should exist between the various realisations of a motive. We will explain the case of motives over $\mathbb{Q}$: a motive over $\mathbb{Q}$ should give rise to a system (M1) to (M10) as below, satisfying axioms (AM1) to (AM5).

Terminology: all the vector spaces are assumed to be of finite dimension; "*almost every* prime number" means "all, except for a finite number."

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(M1). A vector space M_{B} over $\mathbb{Q}$, called the *Betti realisation*.

(M2). A vector space M_{DR} over $\mathbb{Q}$, called the *de Rham realisation*.

(M3). A module $M^{\mathrm{f}}_{\mathbb{A}}$ over $\mathbb{A}^{\mathrm{f}}$, called the *étale cohomology realisation*, which is of finite type, by (M5).

(M4). For almost every prime number p, a vector space $M_{\mathrm{cris}\,p}$ over $\mathbb{Q}_p$, called the *crystalline realisation* of the mod-p reduction.

(M5). Comparison isomorphisms

$$\mathrm{comp}_{\mathrm{DR,B}} : M_{\mathrm{B}} \otimes \mathbb{C} \xrightarrow{\sim} M_{\mathrm{DR}} \otimes \mathbb{C}$$
$$\mathrm{comp}_{\mathbb{A}^{\mathrm{f}},\mathrm{B}} : M_{\mathrm{B}} \otimes \mathbb{A}^{\mathrm{f}} \xrightarrow{\sim} M_{\mathbb{A}}^{\mathrm{f}}$$
$$\mathrm{comp}_{\mathrm{cris}\,p,\mathrm{DR}} : M_{\mathrm{DR}} \otimes \mathbb{Q}_p \xrightarrow{\sim} M_{\mathrm{cris}\,p}$$

(M6). M_{B}, M_{DR}, $M_{\mathbb{A}}^{\mathrm{f}}$, and $M_{\mathrm{cris}\,p}$ are endowed with a finite increasing filtration W, called the *weight filtration*. We also denote by W the filtrations that are induced by extension of scalars. The comparison isomorphisms respect W.

(M7). M_{B} is endowed with an involution F_∞, called the *Frobenius at infinity*, which respects W.

(M8). M_{DR} is endowed with a finite decreasing filtration F, called the *Hodge filtration*. We also denote by F the filtrations that are induced by extension of scalars.

(M9). $M_{\mathbb{A}}^{\mathrm{f}}$ is endowed with an action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ which respects W.

(M10). $M_{\mathrm{cris}\,p}$ is endowed with an automorphism

$$\phi_p : M_{\mathrm{cris}\,p} \to M_{\mathrm{cris}\,p},$$

called the *crystalline Frobenius*, which respects W.

(AM1). M_{B}, endowed with W and with the filtration F of $M_{\mathrm{B}} \otimes \mathbb{C} = M_{\mathrm{DR}} \otimes \mathbb{C}$, is a mixed Hodge $\mathbb{Q}$-structure [**D2?**, Definition 2.3.8].

(AM2). We have two real structures on $M_{\mathrm{B}} \otimes \mathbb{C}$ (identified with $M_{\mathrm{DR}} \otimes \mathbb{C}$ by the comparison isomorphism), namely $M_{\mathrm{B}} \otimes \mathbb{R}$ and $M_{\mathrm{DR}} \otimes \mathbb{R}$; these define antilinear involutions c_{B} and c_{DR}, of which $M_{\mathrm{B}} \otimes \mathbb{R}$ and $M_{\mathrm{DR}} \otimes \mathbb{R}$ are (respectively) the fixed points. These involutions, as well as the linear involution extending F_∞, all commute with one another, and satisfy

$$F_\infty = c_{\mathrm{B}} c_{\mathrm{DR}}.$$

In other words, c_{DR} respects $M_{\mathrm{B}} \subset M_{\mathrm{B}} \otimes \mathbb{C} = M_{\mathrm{DR}} \otimes \mathbb{C}$, and $c_{\mathrm{DR}} | M_{\mathrm{B}} = F_\infty$. *p. 94*

(AM3). For each prime number ℓ, let M_ℓ be given by extension of scalars of $M_{\mathbb{A}}^{\mathrm{f}}$, so that $M_{\mathbb{A}}^{\mathrm{f}}$ is then a restricted product of the M_ℓ. There exists a finite set S of prime numbers such that, for each ℓ, the representation M_ℓ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is unramified outsied of S and ℓ.

(AM4). For large enough S, if $p \notin S$, then, for all $\ell \neq p$, the eigenvalues of a geometric Frobenius at p on the $\mathrm{Gr}_n^W(M_\ell)$, and those of ϕ_p on the $\mathrm{Gr}_n^W(M_{\mathrm{cris}\,p})$, are algebraic numbers whose complex conjugates are all of absolute value $p^{n/2}$, and are ℓ'-adic units for $\ell' \neq p$.

(AM5). Let $c \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be complex conjugation. Then c acts on $M^{\mathrm{f}}_{\mathbb{A}}$ respecting $M_{\mathrm{B}} \subset M^{\mathrm{f}}_{\mathbb{A}}$, and induces the involution F_∞ on M_{B}.

1.5 Remarks. —

i. If M_{DR} is given, then the data of M_{B}, F_∞, and $\mathrm{comp}_{\mathrm{DR},\mathrm{B}}$ satisfying (AM2) is equivalent to that of a new rational structure $M_{\mathrm{B}} \subset M_{\mathrm{DR}} \otimes \mathbb{C}$ that is stable under complex conjugation c_{DR} (set $F_\infty = c_{\mathrm{DR}}|M_{\mathrm{B}}$). By (M6), the filtration W of M_{DR} must remain rational for this new rational structure.

ii. The data of $M^{\mathrm{f}}_{\mathbb{A}}$, $\mathrm{comp}_{\mathbb{A}^{\mathrm{f}},\mathrm{B}}$, and the Galois action, all together, are equivalent to the data of a $\mathbb{Q}_\ell$-vector space M_ℓ for all ℓ, along with a Galois action on M_ℓ and comparison isomorphisms $\mathrm{comp}_{\ell,\mathrm{B}} \colon M_{\mathrm{B}} \otimes \mathbb{Q} \xrightarrow{\sim} M_\ell$. We have to assume the existence of a lattice $L \subset M_{\mathrm{B}}$ such that the $\mathrm{comp}_{\ell,\mathrm{B}}(L \otimes \mathbb{Z}_\ell)$ are Galois stable. We define $M^{\mathrm{f}}_{\mathbb{A}}$ from the M_ℓ as the restricted product of the M_ℓ with respect to the $\mathrm{comp}_{\ell,\mathrm{B}}(L \otimes \mathbb{Z}_\ell)$ for an arbitrary lattice L: this restricted product is Galois stable, and the $\mathrm{comp}_{\ell,\mathrm{B}}$ induce $\mathrm{comp}_{\mathbb{A}^{\mathrm{f}},\mathrm{B}}$.

   The data of M_ℓ, $\mathrm{comp}_{\ell,\mathrm{B}}$, and the Galois action (resp. $M^{\mathrm{f}}_{\mathbb{A}}$, $\mathrm{comp}_{\mathbb{A}^{\mathrm{f}},\mathrm{B}}$, and the action), all together, are also equivalent to the data of a Galois action on $M_{\mathrm{B}} \otimes \mathbb{Q}_\ell$ (resp. $M_{\mathrm{B}} \otimes \mathbb{A}^{\mathrm{f}}$). By (M6) and (M9), the filtration of $M_{\mathrm{B}} \otimes \mathbb{Q}_\ell$ (resp. $M_{\mathrm{B}} \otimes \mathbb{A}^{\mathrm{f}}$) induced by W must be stable under $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

iii. If M_{DR} is given, then the data of $M_{\mathrm{cris}\,p}$, along with its crystalline Frobenius and $\mathrm{comp}_{\mathrm{cris}\,p,\mathrm{DR}}$, is equivalent to the data of an automorphism ϕ_p of $M_{\mathrm{DR}} \otimes \mathbb{Q}_p$. By (M6) and (M10), the filtration of $M_{\mathrm{DR}} \otimes \mathbb{Q}_p$ induced by W must be stable under ϕ_p.

We will often tacitly use these remarks to describe a system (M1)–(M10).

1.6. A scheme X of finite type over $\mathbb{Q}$ should define, for each i, a motive $M := \mathrm{H}^i_{\mathrm{mot}}(X)$. In this section, we will partially describe the system (M1)–(M10) of realisations of M in the case where X is separated and smooth over $\mathbb{Q}$. *p. 95*

We have $M_{\mathrm{B}} = \mathrm{H}^i(X(\mathbb{C}), \mathbb{Q})$, and F_∞ is induced by the complex conjugation of $X(\mathbb{C})$; $M_{\mathrm{DR}} = \mathrm{H}^i_{\mathrm{DR}}(X) := \mathbb{H}^i(X, \Omega^\bullet_X)$, and the Hodge filtration that that of the mixed Hodge theory [**D2?**, Section 3.2]; $M_\ell = \mathrm{H}^i(X \otimes \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$ is the ℓ-adic étale cohomology of the scheme over $\overline{\mathbb{Q}}$ induced from X by extension of scalars, and the action of $\mathrm{Gal}(\overline{\mathbb{Q}}, \mathbb{Q})$ is given by structure transport. *Notation:* $X \otimes \overline{\mathbb{Q}}$, cf. (0.4).

Suppose that X is smooth and proper, and let S be a finite set of prime numbers such that X is the general fibre of $X^\sim$, which is smooth and proper over $\mathrm{Spec}(\mathbb{Z}) \setminus S$. For $p \notin S$, $M_{\mathrm{cris}\,p}$ is the crystalline cohomology of the reduction $X^\sim \otimes \mathbb{F}_p$ of X modulo p, tensored over $\mathbb{Z}_p$ with $\mathbb{Q}_p$. The crystalline Frobenius ϕ_p is induced by the inverse image morphism of the Frobenius $\mathrm{Fr} \colon X^\sim \otimes \mathbb{F}_p \to X^\sim \otimes \mathbb{F}_p$.

More generally, suppose that we have some smooth and proper $\overline{X}$ over $\mathrm{Spec}(\mathbb{Z}) \setminus S$, as well as a relative normal crossing divisor D; let X be the general fibre of $\overline{X} \setminus D$. Then the realisation $M_{\mathrm{cris}\,p}$ is defined for $p \notin S$; its most natural definition is given by the generalisation of the crystalline theory, considered by Faltings in [**Fa?**, IV], to the "logarithmic poles" case.

The comparison isomorphism $\mathrm{comp}_{\mathrm{DR,B}}$ is (1.1.3), and the comparison isomorphism $\mathrm{comp}_{\ell,\mathrm{B}}$ is (1.1.4).

In the smooth and proper case, the comparison isomorphism $\mathrm{comp}_{\mathrm{cris}\,p,\mathrm{DR}}$ comes from §7.26 of (P. Berthelot and A. Ogus, *Notes on crystalline cohomology*, Princeton University Press and Tokyo University Press, 1978). For the general case, see [**Fa?**, IV]. Finally, the weight filtration W is that of the mixed Hodge theory from [**D2?**, Section 3.2]. See also [**D5?**].

1.7. An additional data that we have on the cohomology $M := \mathrm{H}^i_{\mathrm{mot}}(X)$ when X is smooth over $\mathbb{Q}$ is that of a comparison isomorphism, for almost all p, in the sense of Fontaine–Messing (cf. [**FM?**,**Fa?**]), relating M_p, endowed with the action of a decomposition group of p, to $M_{\mathrm{DR}} \otimes \mathbb{Q}_p$, endowed with its Hodge filtration and its crystalline Frobenius.

For all p, we should also have a "crystalline" structure of the following type.

- **Semi-stable case.** Let T_p be the Zariski tangent space of $\mathrm{Spec}(\mathbb{Z}_p)$ at its closed point. We complete it to a projective line $\overline{T}_p$ over $\mathbb{F}_p$, and we can lift $(\overline{T}_p, 0, \infty)$ to a projective line endowed with two marked points over $\mathbb{Z}_p$: $(\overline{T}_p^{\sim}, 0, \infty)$. We want an F-isocrystal with logarithmic poles on $(\overline{T}_p, 0, \infty)$ (cf. [**Fa?**]). Such an object induces, on $\overline{T}_p^{\sim} \otimes \mathbb{Q}_p$, a module with connection $\mathcal{V}$ with logarithmic poles at 0 and at ∞, and we want for the residue of the connection at 0 and at ∞ to be nilpotent. If φ is a section of $\overline{T}_p^{\sim}$, over $\mathrm{Spec}(\mathbb{Z}_p)$, with derivative equal to 1 at the closed point, then $\varphi^*\mathcal{V}$ is independent of the choice of φ, and $\mathrm{comp}_{\mathrm{DR,cris}\,p}$ should then be identified with the de Rham realisation $\otimes\mathbb{Q}_p$. *p. 96*
- **General case.** The data of the previous type, over a large-enough finite Galois extension E of $\mathbb{Q}_p$ that is $\mathrm{Gal}(E/\mathbb{Q}_p)$-equivariant.

A Fontaine–Messing comparison isomorphism should again link this object and M_p endowed with the action of a decomposition group of p.

1.8 Variant. We should also have (M1)–(M10) for $M := \mathrm{H}^i_{\mathrm{mot}}(X)$, where X is not necessarily smooth. The crystalline data pose a problem.

We would also like to have (M1)–(M10) for cohomology with proper support.

1.9 Definition. A *realisation system* is a system (M1)–(M10) that satisfies (A1)–(A5).