Hodge Theory II

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Translator's note

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Introduction

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Work presented as a doctoral thesis at l'Université d'Orsay.

By Hodge, the cohomology space $\mathrm{H}^n(X,\mathbb{C})$ of a compact Kähler variety X is endowed with a "Hodge structure" of weight n, i.e. a natural bigrading

$$\mathrm{H}^n(X,\mathbb{C}) = \bigoplus_{p+q=n} \mathrm{H}^{p,q}$$

that satisfies $\overline{\mathrm{H}^{p,q}} = \mathrm{H}^{q,p}$. We will show here that the complex cohomology of a non-singular, not necessarily compact, algebraic variety is endowed with a structure of a slightly more general type, which presents $\mathrm{H}^n(X,\mathbb{C})$ as a "successive extension" of Hodge structures of decreasing weights, contained between $2n$ and n, whose Hodge numbers $h^{p,q} = \dim \mathrm{H}^{p,q}$, are zero for both $p > n$ and $q > n$.

The reader will find an explanation in [5] of the yoga that underlies this construction. *p. 6*

The proof, which is essentially algebraic, relies on one hand on Hodge theory, and on the other on Hironaka's resolution of singularities, which allows us, via a spectral sequence, "to express" the cohomology of a non-singular quasi-projective algebraic variety in terms of the cohomology of non-singular projective varieties.

Section 1 contains, apart from reminders on filtrations gathered together for the ease of the reader, two key results:

a) Theorem 1.2.10, which will only be used via its corollary, Theorem 2.3.5, which gives the fundamental properties of "mixed Hodge structures."
b) The "two filtrations lemma," Lemma 1.3.16.

Section 2 recalls Hodge theory and introduces mixed Hodge structures.

The heart of this work is §3.2, which defines the mixed Hodge structure of $\mathrm{H}^n(X,\mathbb{C})$, and establishes some degenerations of spectral sequences.

Section 4 gives diverse applications, all following from Theorem 4.1.1 and the theory of the (K/k)-trace, for the resulting Hodge structures (Corollary 4.1.2). The principal ones are Theorem 4.2.6 and Corollary 4.4.15.

1 Filtrations

1.1 Filtered objects

1.1.1. Let $\mathscr{A}$ be an abelian category.

We will be considering $\mathbb{Z}$-filtrations, finite, in general, on the objects of $\mathscr{A}$:

1.1.2. A *decreasing* (resp. *increasing*) *filtration* F of an object A of $\mathscr{A}$ is a family $(F^n(A))_{n\in\mathbb{Z}}$ (resp. $(F_n(A))_{n\in\mathbb{Z}}$) of sub-objects of A satisfying

$$\forall n,m \quad n \leqslant m \implies F^m(A) \subset F^n(A)$$

(resp. $n \leqslant m \implies F_n(A) \subset F_m(A)$).

A *filtered object* is an object endowed with a filtration. When there is no chance of confusion, we often denote by the same letter filtrations on different objects of $\mathscr{A}$.

If F is a decreasing (resp. increasing) filtration on A, then we set $F^\infty(A) = 0$ and $F^{-\infty}(A) = A$ (resp. $F_{-\infty}(A) = 0$ and $F_\infty(A) = A$).

The *shifted filtrations* of a decreasing filtration W are defined by

$$W[n]^p(A) = W^{n+p}(A)$$

for $n \in \mathbb{Z}$.

1.1.3. If R is a decreasing (resp. increasing) filtration of A, then the $F_n(A) = F^{-n}(A)$ (resp the $F^n(A) = F_{-n}(A)$) form an increasing (resp. decreasing) filtration of A. This allows us in principal to consider only decreasing filtrations; *unless otherwise explicitly mentioned, when we say "filtration" we always mean "decreasing filtration"*. *p. 7*

1.1.4. A filtration F of A is said to be *finite* if there exist n and m such that $F^n(A) = A$ and $F^m(A) = 0$.

1.1.5. A *morphism* from a filtered object (A,F) to a filtered object (B,F) is a morphism f from A to B that satisfies $f(F^n(A)) \subset F^n(B)$ for all $n \in \mathbb{Z}$.

Filtered objects (resp. finite filtered objects) of $\mathscr{A}$ form an additive category in which inductive limits and finite projective limits exist (and thus kernels, cokernels, images, and coimages of a morphism).

A morphism $f:(A,F)\to(B,F)$ is said to be *strict*, or *strictly compatible with the filtrations*, if the canonical arrows from $\mathrm{Coim}(f)$ to $\mathrm{Im}(f)$ is an isomorphism of filtered objects (cf. (1.1.11)).

1.1.6. Let $(-)^\circ$ be the contravariant identity functor from $\mathscr{A}$ to the dual category $\mathscr{A}^\circ$. If (A,F) is a filtered object of $\mathscr{A}$, then the $(A/F^n(A))^\circ$ can be identified with sub-objects of A°. The *dual* filtration on A° is defined by

$$F^n(A^\circ)=(A/F^{1-n})^\circ.$$

The double dual of (A,F) can be identified with (A,F). This construction identifies the dual of the category of filtered objects of $\mathscr{A}$ with the category of filtered objects of $\mathscr{A}^\circ$.

1.1.7. If (A,F) is a filtered object of $\mathscr{A}$, then its *associated graded* is the object of $\mathscr{A}^{\mathbb{Z}}$ defined by

$$\mathrm{Gr}^n(A)=F^n(A)/F^{n+1}(A).$$

The convention (1.1.6) is justified by the simple formula

$$\mathrm{Gr}^n(A^\circ)=\mathrm{Gr}^{-n}(A)^\circ$$

which follows from the self-dual diagram **!!TO-DO: diagram (1.1.7.1)!!**

1.1.8. Let (A,F) be a filtered object, and $j:X\hookrightarrow A$ a sub-object of A. The *filtration induced by F* (or simply *induced filtration*) on X is the unique filtration on X such that j is strictly compatible with the filtrations; we have

$$F^n(X)=j^{-1}(F^n(A))=X\cap F^n(A).$$

Dually, the *quotient filtration* on A/X (the unique filtration such that $p:A\to A/X$ is strictly compatible with the filtrations) is given by

$$F^n(A/X)=p(F^n(A))\cong(X+F^n(A))/X\cong F^n(A)/(X\cap F^n(A)).$$

Lemma 1.1.9. *If X and Y are sub-objects of A, with $X\subset Y$, then on $Y/X\xrightarrow{\sim}\mathrm{Ker}(A/X\to A/Y)$ the quotient filtration of Y agrees with that induced by that of A/X. In the diagram* ***!!TO-DO: diagram!!*** *the arrows are strict.* *p. 8*

1.1.10. We call the filtration (1.1.9) on Y/X the *filtration induced by that of A* (or simply the *induced filtration*). By (1.1.9), its definition is self-dual.

In particular, if $\Sigma: A\xrightarrow{f}B\xrightarrow{G}C$ is a **!!TO-DO: o-suite?!!** sequence, and if B is filtered, then $\mathrm{H}(\Sigma)=\mathrm{Ker}(g)/\mathrm{Im}(f)=\mathrm{Ker}(\mathrm{Coker}(f)\to\mathrm{Coim}(g))$ is endowed with a canonical induced filtration.

The reader can show that:

Proposition 1.1.11.

—

i. *Let $f:(A,F)\to(B,F)$ be a morphism of filtered objects with finite filtrations. For f to be strict, it is necessary and sufficient that the sequence*

$$0\to \mathrm{Gr}(\mathrm{Ker}(f))\to \mathrm{Gr}(A)\to \mathrm{Gr}(B)\to \mathrm{Gr}(\mathrm{Coker}(f))\to 0$$

be exact.

ii. *Let $\sigma:(A,F)\to(B,F)\to(C,F)$ be a* ***!!TO-DO: o-suite?!!*** *sequence of strict morphisms. We then have*

$$\mathrm{H}(\mathrm{Gr}(\Sigma))\cong \mathrm{Gr}(\mathrm{H}(\Sigma))$$

canonically. In particular, if Σ is exact in $\mathscr{A}$, then $\mathrm{Gr}(\Sigma)$ is exact in $\mathscr{A}^{\mathbb{Z}}$.

In a category of modules, to say that a morphism $f:(A,F)\to(B,F)$ is strict implies that every $b\in B$ of filtration $\geqslant n$ (i.e. $b\in F^n(B)$) that is in the image of A is already in the image of $F^n(A)$:

$$f(F^n(A))=f(A)\cap F^n(B).$$

1.1.12. If $\otimes:\mathscr{A}_1\times\ldots\times\mathscr{A}_n\to\mathscr{B}$ is a multiadditive right-exact functor, and if A_i is an object of finite filtration of $\mathscr{A}_i$ (for $1\leqslant i\leqslant n$), then we define a filtration on $\bigotimes_{i=1}^n A_i$ by

$$F^k(\bigotimes_{i=1}^n A_i)=\sum_{\sum k_i=k}\mathrm{Im}(\bigotimes_{i=1}^n F^{k_i}(A_i)\to\bigotimes_{i=1}^n A_i)$$

(a sum of sub-objects).

Dually, if H is left-exact, then we set

$$F^k(H(A_i))=\bigcap_{\sum k_i=k}\mathrm{Ker}(H(A_i)\to H(A_i/F^{k_i}(A_i))).$$

If H is exact, then the two definitions are equivalent. *p. 9*

We extend these definitions to contravariant functors in certain variables by (1.1.6). In particular, for the left-exact functor Hom, we set

$$F^k(\mathrm{Hom}(A,B))=\{f:A\to B\mid f(F^n(A))\subset F^{n+k}(B)\ \forall n\}.$$

We thus have

$$\mathrm{Hom}((A,F),(B,F))=F^0(\mathrm{Hom}(A,B)).$$

Under the above hypotheses, we have obvious morphisms

$$\bigotimes_{i=1}^n \mathrm{Gr}(A_i)\to\mathrm{Gr}(\bigotimes_{i=1}^n A_i)$$
$$\mathrm{Gr}\,H(A_i)\to H(\mathrm{Gr}(A_i)).$$

If H is exact, then these are isomorphisms and inverse to one another.

These constructions are compatible with composition of functors, in a sense whose details we leave to the reader.

1.2 Opposite filtrations

1.2.1. Let A be an object of $\mathscr{A}$ endowed with filtrations F and G. By definition, $\mathrm{Gr}_F^n(A)$ is a quotient of a sub-object of A and, as such, is endowed with a filtration induced by G (1.1.10). Passing to the associated graded defines a bigraded object $(\mathrm{Gr}_G^n \mathrm{Gr}_F^m(A))_{n,m\in\mathbb{Z}}$. By a lemma of Zassenhaus, $\mathrm{Gr}_G^n \mathrm{Gr}_F^m(A)$ and $\mathrm{Gr}_F^m \mathrm{Gr}_G^n(A)$ are canonically isomorphic: if we define the induced filtrations (1.1.10) as quotient filtrations of the induced filtrations on a sub-object, then we have

$$\mathrm{Gr}_G^n \mathrm{Gr}_F^m(A) \cong (F^m(A)\cap G^n(A))/((F^{m+1}(A)\cap G^n(A)) + (F^m(A)\cap G^{n+1}(A)))$$

$$=$$

$$\mathrm{Gr}_F^m \mathrm{Gr}_G^n(A) \cong (G^n(A)\cap F^m(A))/((G^{n+1}(A)\cap F^m(A)) + (G^n(A)\cap F^{m+1}(A)))$$

1.2.2. Let H be a third filtration of A. It induces a filtration on $\mathrm{Gr}_F(A)$, and thus on $\mathrm{Gr}_G \mathrm{Gr}_F(A)$. It also induces a filtration on $\mathrm{Gr}_F \mathrm{Gr}_G(A)$. We note that these filtrations do not in general correspond to one another under the isomorphism (1.2.1). In the expression $\mathrm{Gr}_H \mathrm{Gr}_G \mathrm{Gr}_F(A)$, G and H thus play a symmetric role, but not F and G.

1.2.3. Two *finite* filtrations F and $\overline{F}$ on A are said to be *n-opposite* if $\mathrm{Gr}_F^p \mathrm{Gr}_{\overline{F}}^q(A) = 0$ for $p+q \neq n$.

1.2.4. If $A^{p,q}$ is a bigraded object of $\mathscr{A}$ such that

a. $A^{p,q} = 0$ except for a finite number of pairs (p,q), and
b. $A^{p,q} = 0$ for $p+q \neq n$

then we define two n-opposite filtrations of $A = \sum_{p,q} A^{p,q}$ by setting p. 10

$$F^p(A) = \sum_{p'\geqslant p} A^{p',q'} \tag{1.2.4.1}$$

$$\overline{F}^q(A) = \sum_{q'\geqslant q} A^{p',q'}. \tag{1.2.4.2}$$

We have

$$\mathrm{Gr}_F^p \mathrm{Gr}_{\overline{F}}^q(A) = A^{p,q}. \tag{1.2.4.3}$$

Conversely:

Proposition 1.2.5.

—

i. Let F and $\overline{F}$ be finite filtrations on A. For F and $\overline{F}$ to be n-opposite, it is necessary and sufficient that, for all p,q,

$$[p+q = n+1] \implies [F^p(A) \oplus \overline{F}^q(A) \xrightarrow{\sim} A].$$

ii. If F and $\overline{F}$ are n-opposite, and if we set

$$\begin{cases} A^{p,q} = 0 & \text{for } p+q \neq n \\ A^{p,q} = F^p(A) \cap \overline{F}^q(A) & \text{for } p+q = n \end{cases}$$

then A is the direct sum of the $A^{p,q}$, and F and $\overline{F}$ come from the bigrading $A^{p,q}$ of A by the procedure of (1.2.4).

!!TO-DO: why is the following proof not appearing in the PDF version?!!

1.2.6. The constructions (1.2.4) and (1.2.5) establish equivalences of categories that are quasi-inverse to one another between objects of $\mathscr{A}$ endowed with two finite n-opposite filtrations and bigraded objects of $\mathscr{A}$ of the type considered in (1.2.4).

Definition 1.2.7. *Three* finite *filtrations W, F, and $\overline{F}$ on A are said to be* opposite *if* p. 11

$$\mathrm{Gr}_F^p \mathrm{Gr}_{\overline{F}}^q \mathrm{Gr}_W^n(A) = 0$$

for $p+q+n \neq 0$.

This condition is symmetric in F and $\overline{F}$. It implies that F and $\overline{F}$ induce on $W^n(A)/W^{n+1}(A)$ two $(-n)$-opposite filtrations. We set

$$A^{p,q} = \mathrm{Gr}_F^p \mathrm{Gr}_{\overline{F}}^q \mathrm{Gr}_F^{-p-q}(A)$$

whence decompositions (1.2.4), (1.2.5)

$$W^n(A)/W^{n+1}(A) = \bigoplus_{p+q=-n} A^{p,q} \tag{1.2.7.1}$$

which makes $\mathrm{Gr}_W(A)$ into a bigraded object.

Lemma 1.2.8. *Let W, F, and $\overline{F}$ be three finite opposite filtrations, and σ a sequence $(p_i, q_i)_{i\geqslant 0}$ pairs of integers satisfying*

a. $p_i \leqslant p_j$ and $q_i \leqslant q_j$ for $i \geqslant j$, and
b. $p_i + q_i = p_0 + q_0 - i + 1$ for $i > 0$.

Set $p = p_0$, $q = q_0$, $n = -p-q$, and

$$A_\sigma = \left(\sum_{0\leqslant i}(W^{n+i}(A) \cap F^{p_i}(A))\right) \cap \left(\sum_{0\leqslant i}(W^{n+i}(A) \cap \overline{F}^{q_i}(A))\right).$$

Then the projection from $W^n(A)$ to $\mathrm{Gr}_W^n(A)$ induces an isomorphism

$$A_\sigma \xrightarrow{\sim} A^{p,q} \subset \mathrm{Gr}_W^n(A).$$

Proof. We will prove by induction on k the following claim:

$(*_k)$ The projection from $W^n(A)/W^{n+k}$ to $\mathrm{Gr}^n_W(A)$ induces an isomorphism from

$$\left(\left(\sum_{i<k}(W^{n+i}(A)\cap F^{p_i}(A))+W^{n+k}(A)\right)\cap\left(\sum_{i<k}(W^{n+i}(A)\cap\overline{F}^{q_i}(A))+W^{n+k}(A)\right)\right)/W^{n+k}(A)$$

to $A^{p,q}\subset\mathrm{Gr}^n_W(A)$.

For $k=1$, this is exactly the definition of $A^{p,q}$.

By (1.2.5), (i) we have

$$F^{p_k}(\mathrm{Gr}^{n+k}_W(A))\oplus\overline{F}^{q_k}(\mathrm{Gr}^{n+k}_W(A))\xrightarrow{\sim}\mathrm{Gr}^{n+k}_W(A). \tag{1.2.8.1}$$

Set

$$\begin{aligned}
B&=\sum_{i<k}(W^{n+i}(A)\cap F^{p_i}(A))\\
C&=\sum_{i<k}(W^{n+i}(A)\cap\overline{F}^{q_i}(A))\\
B'&=(W^{n+k}(A)\cap F^{p_k}(A))+W^{n+k+1}(A)\\
C'&=(W^{n+k}(A)\cap\overline{F}^{q_k}(A))+W^{n+k+1}(A)\\
D&=W^{n+k}(A)\\
E&=W^{n+k+1}(A).
\end{aligned}$$

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Equation (1.2.8.1) can then be written as

$$\begin{aligned}B'+C'&=D\\ B'\cap C'&=E.\end{aligned}$$

We also have, since $p_k\leqslant p_i$ (for $i\leqslant k$),

$$B\cap D\subset F^{p_k}(A)\cap W^{n+k}(A)\subset B'$$

and, since $q_k\leqslant q_i$ (for $i\leqslant k$),

$$C\cap D\subset\overline{F}^{q_k}(A)\cap W^{n+k}(A)\subset C'.$$

The claim $(*_{k+1})$ and then follows from $(*_k)$ and the following lemma. ☐

Lemma 1.2.9. *Let B, C, B', C', D, and E be sub-objects of A. Suppose that*

$$\begin{aligned}B'+C'&=D &\qquad B'\cap C'&=E\\ B\cap D&\subset B' &\qquad C\cap D&\subset C'.\end{aligned}$$

Then

$$((B+B')\cap(C+C'))/E\xrightarrow{\sim}((B+D)\cap(C+D))/D.$$

Proof. To prove surjectivity, we write

$$\begin{aligned}((B+B')\cap(C+C'))+D &= (((B+B')\cap(C+C'))+B')+C' \\ &= ((B+B')\cap(C+C'+B'))+C' \\ &= (B+B'+C')\cap(C+C'+B') \\ &= (B+D)\cap(C+D).\end{aligned}$$

To prove injectivity, we write

$$(B+B')\cap(C+C')\cap D = ((B+B')\cap D)\cap((C+C')\cap D).$$

Since $B' \subset D$, we have

$$\begin{aligned}(B+B')\cap D &= (B\cap D)+B' \\ &= B'\end{aligned}$$

and similarly

$$(C+C')\cap D = C'$$

and

$$\begin{aligned}(B+B')\cap(C+C')\cap D &= B'\cap C' \\ &= E.\end{aligned}$$

□

This finishes the proof of (1.2.8), noting that (1.2.8) is equivalent to $(*_k)$ for large k.

Theorem 1.2.10.
Let $\mathscr{A}$ be an abelian category, and provisionally denote by $\mathscr{A}'$ the category of objects of $\mathscr{A}$ endowed with three opposite filtrations W, F, and $\overline{F}$. The morphisms in $\mathscr{A}'$ are the morphisms of $\mathscr{A}$ that are compatible with the three filtrations.

- *i.* *$\mathscr{A}'$ is an abelian category.*
- *ii.* *The kernel (resp. cokernel) of an arrow $f : A \to B$ in $\mathscr{A}'$ is the kernel (resp. cokernel) of f in $\mathscr{A}$, endowed with the filtrations induced by those of A (resp. the quotients of those of B).*
- *iii.* *Every morphism $f : A \to B$ in $\mathscr{A}'$ is strictly compatible with the filtrations W, F, and $\overline{F}$; the morphism $\mathrm{Gr}_W(f)$ is compatible with the bigradings of $\mathrm{Gr}_W(A)$ and $\mathrm{Gr}_W(B)$; the morphisms $\mathrm{Gr}_F(f)$ and $\mathrm{Gr}_{\overline{F}}(f)$ are strictly compatible with the filtration induced by W.*
- *iv.* *The "forget the filtrations" functors, Gr_W, Gr_F, and $\mathrm{Gr}_{\overline{F}}$, and*

$$\begin{aligned}\mathrm{Gr}_W \mathrm{Gr}_F &\simeq \mathrm{Gr}_F \mathrm{Gr}_W \\ &\simeq \mathrm{Gr}_{\overline{F}} \mathrm{Gr}_F \mathrm{Gr}_W \\ &\simeq \mathrm{Gr}_{\overline{F}} \mathrm{Gr}_W \simeq \mathrm{Gr}_W \mathrm{Gr}_{\overline{F}}\end{aligned}$$

from $\mathscr{A}'$ to $\mathscr{A}$ are exact.

Denote by $\sigma_0(p,q)$ and $\sigma_1(p,q)$ the sequences *p. 13*

$$\begin{aligned}\sigma_0(p,q) &= (p,q),(p,q),(p,q-1),(p,q-2),(p,q-3),\ldots \\ \sigma_0(p,q) &= (p,q),(p,q),(p-1,q),(p-2,q),(p-3,q),\ldots\end{aligned}$$

and, with the notation of (1.2.8), set

$$A_i^{p,q} = A_{\sigma_i(p,q)} \qquad \text{(for } i = 0,1\text{).}$$

If $f : A \to B$ is compatible with W, F, and $\overline{F}$, then we have

$$f(A_i^{p,q}) \subset B_i^{p,q} \qquad \text{(for } i = 0,1\text{).} \tag{1.2.10.1}$$

Claim (iii) then follows from the following lemma:

Lemma 1.2.11. *The $A_i^{p,q}$ give a bigrading of A. We have*

$$W^n(A) = \sum_{n+p+q \leqslant 0} A_i^{p,q} \qquad \textit{(for } i = 0,1\textit{)} \tag{1.2.11.1}$$

$$F^p(A) = \sum_{p' \geqslant p} A_0^{p',q'} \tag{1.2.11.2}$$

$$\overline{F}^q(A) = \sum_{q' \geqslant q} A_1^{p',q'}. \tag{1.2.11.3}$$

Proof. By symmetry, it suffices to prove the claims concerning $i = 0$. Set $A_0 = \bigoplus A_0^{p,q}$ and define filtrations W and F on A_0 by the equations of (1.2.11). The canonical map i from A_0 to A is compatible with the filtrations W and F. Furthermore, by (1.2.8), $\mathrm{Gr}_W(i)$ is an isomorphism, and induces isomorphisms of graded objects

$$\sum_{p+q=n} A_0^{p,q} \xrightarrow{\sim} \mathrm{Gr}_W^{-n}(A) = \sum_{p+q=n} A^{p,q}. \tag{1.2.11.4}$$

The morphism i is thus an isomorphism, and the $A_0^{p,q}$ give a bigrading of A.

Equation (1.2.11.1) then says that $\mathrm{Gr}_W(i)$ is an isomorphism. By (1.2.11.4), $\mathrm{Gr}_F \mathrm{Gr}_W(i)$ is an isomorphism, and thus so too are $\mathrm{Gr}_W \mathrm{Gr}_F(i)$ and $\mathrm{Gr}_F(i)$. Equation (1.2.11.2) then follows. □

1.2.12. We now prove (1.2.10). Let $f : A \to B$ in $\mathscr{A}'$ and endow $K = \mathrm{Ker}(f)$ with the filtrations induced by those of A. By (1.2.11), $\mathrm{Gr}_W(K) \hookrightarrow \mathrm{Gr}_W(A)$; furthermore, the filtration F (resp. $\overline{F}$) on K induces on $\mathrm{Gr}_W(K)$ the inverse image filtration of the filtration F on $\mathrm{Gr}_W(A)$. The sub-object $\mathrm{Gr}_W(K)$ of $\mathrm{Gr}_W(A)$ is then compatible with the bigrading of $\mathrm{Gr}_W(A)$:

$$\mathrm{Gr}_W(K) = \bigoplus_{p,q} (\mathrm{Gr}_W(K) \cap A^{p,q}).$$

We thus deduce that

$$\mathrm{Gr}_F^p \mathrm{Gr}_{\overline{F}}^q \mathrm{Gr}_W^n(K) \hookrightarrow \mathrm{Gr}_F^p \mathrm{Gr}_{\overline{F}}^q \mathrm{Gr}_W^n(A);$$

the filtrations of W, F, and $\overline{F}$ on K are thus opposite, and K is a kernel of f in $\mathscr{A}'$. This, combined with the dual result, proves (ii). *p. 14*

If f is an arrow of $\mathscr{A}'$, then the canonical morphism from Coim(f) to Im(f) is an isomorphism in $\mathscr{A}$; by (iii), it is also an isomorphism in $\mathscr{A}'$, which is thus abelian.

The "forget the filtrations" functor is exact by (ii). The exactness of the other functors in (iv) follows immediately from (ii), (iii), and (1.1.11), (i) or (ii).

1.2.13. Let A be an object of $\mathscr{A}$ endowed with a finite *increasing* filtration $W_\bullet$, and two finite decreasing filtrations F and $\overline{F}$. The construction (1.1.3) associates to $W_\bullet$ a decreasing filtration $W^\bullet$. We say that the filtrations $W_\bullet$, F, and $\overline{F}$ are *opposite* if the filtrations $W^\bullet$, F, and $\overline{F}$ are, i.e. if, for all n, the filtrations induced by F and $\overline{F}$ on

$$\mathrm{Gr}_n^W(A) = W_n(A)/W_{n-1}(A)$$

are n-opposite.

Theorem (1.2.10) translates trivially to this variation.

1.3 The two filtrations lemma

1.3.1. Let K be a differential complex of objects of $\mathscr{A}$, endowed with a filtration F. The filtration is said to be *biregular* if it induces a finite filtration on each component of K.

We recall the definition of the terms $E_r^{pq}(K,F)$, or simply E_r^{pq}, of the spectral sequence defined by F. We set

$$Z_r^{pq} = \mathrm{Ker}(d\colon F^p(K^{p+q}) \to F^{p+q+1}/F^{p+r}(K^{p+q+1}))$$

and dually we define B_r^{pq} by the formula

$$K^{p+q}/B_r^{pq} = \mathrm{Coker}(d\colon F^{p-r+1}(K^{p+q+1}) \to K^{p+q}/F^{p+1}(K^{p+q})).$$

These formulas still make sense for $r = \infty$.

We note that the use here of the notation B_r^{pq} is different to that of Godement [6].

We have, by definition:

$$E_r^{pq} = \mathrm{Im}(Z_r^{pq} \to K^{p+q}/B_r^{pq}) \tag{1.3.1.1}$$

$$= Z_r^{pq}/(B_r^{pq} \cap Z_r^{pq}) \tag{1.3.1.2}$$

$$= \mathrm{Ker}(K^{p+q}/B_r^{pq} \to K^{p+q}/(Z_r^{pq} + B_r^{pq})). \tag{1.3.1.3}$$

We can thus write

$$\begin{aligned} B_r^{p\bullet} \cap Z_r^{p\bullet} &:= (dF^{p-r+1} + F^{p+1}) \cap (d^{-1}F^{p+r} \cap F^p) \\ &= (dF^{p-r+1} \cap F^p) + (F^{p+1} \cap d^{-1}F^{p+r}) \end{aligned} \tag{1.3.1.4}$$

since $dF^{p-r+1} \subset d^{-1}F^{p+r}$ and $F^{p+1} \subset F^p$. p. 15

For $r < \infty$, the E_r form a complex graded by the degree $p - r(p+q)$, and E_{r+1} can be expressed as the cohomology of this complex:

$$E_{r+1}^{pq} = \mathrm{H}(E_r^{p-r,q+r-1} \xrightarrow{d_r} E_r^{pq} \xrightarrow{d_r} E_r^{p+r,q-r+1}). \tag{1.3.1.5}$$

For $r = 0$, we have

$$E_0^{\bullet\bullet} = \mathrm{Gr}_F^{\bullet}(K^{\bullet}). \tag{1.3.1.6}$$

Proposition 1.3.2.

Let K be a complex endowed with a biregular filtration F. The following conditions are equivalent:

i. The spectral sequence defined by F degenerates ($E_1 = E_\infty$).
ii. The morphisms $d: K^i \to K^{i+1}$ are strictly compatible with the filtrations.

Proof. We will prove this in the case where $\mathscr{A}$ is a category of modules. For fixed p and q, the hypothesis that the arrows d_r with domains E_r^{pq} be zero for $r \geqslant 1$ implies that, if $x \in F^p(K^{p+q})$ satisfies $dx \in F^{p+1}(K^{p+q+1})$, then there exists $y \in K^{p+q}$ such that $dy = 0$ and such that x and y have the same image in E_1^{pq}. Modifying y by a boundary, and setting $z = x - y$, we then have

$$\forall x \in F^p(K^{p+q}) \left[dx \in F^{p+1}(K^{p+q+1}) \implies \exists z \text{ s.t. } z \in F^{p+1}(K^{p+q}) \text{ and } dz = dx \right]$$

or, in other words,

$$F^{p+1}(K^{p+q+1}) \cap dF^p(K^{p+q}) = dF^{p+1}(K^{p+q}). \tag{1}$$

If this condition is satisfied for arbitrary p and q, then by induction on r we have

$$F^{p+r} \cap dF^p = dF^{p+r}$$

which, for large $p + r$, can be written as

$$F^p \cap dK = dF^p. \tag{2}$$

Claim (2) trivially implies (1), and is equivalent to (ii), which proves the proposition. □

1.3.3. If (K, F) is a filtered complex, we denote by $\mathrm{Dec}(K)$ the complex K endowed with the *shifted filtration*

$$\mathrm{Dec}(F)^p K^n = Z_1^{p+n,-p}.$$

This filtration is compatible with the differentials:

$$\begin{aligned} dZ_1^{p+n,-p} &\subset F^{p+n+1}(K^{n+1}) \cap \mathrm{Ker}(d) \\ &\subset Z_\infty^{p+n+1,-p} \\ &\subset Z_1^{p+n+1,-p}. \end{aligned}$$

Since

$$\begin{aligned} Z_1^{p+1+n,-p-1} &\subset F^{p+1+n}(K^n) \\ &\subset B_1^{p+n,-p} \\ &\subset Z_1^{p+n,-p} \end{aligned} \tag{1.3.3.1}$$

the evident arrow from $Z_1^{p+n,-p}/Z_1^{p+1+n,-p-1}$ to $Z_1^{p+n,-p}/B_1^{p+n,-p}$ is a morphism

$$u : E_0^{p,n-p}(\mathrm{Dec}(K)) \to E_1^{p+n,-p}(K). \tag{1.3.3.2}$$

Proposition 1.3.4.

—

i. The morphisms (1.3.3.2) form a morphism of graded complexes from $E_0(\mathrm{Dec}(K))$ *to* $E_1(K)$.

ii. This morphism induces an isomorphism on cohomology.

iii. It induces step-by-step (via (1.3.1.5)) isomorphisms of graded complexes $E_r(\mathrm{Dec}(K)) \xrightarrow{\sim} E_{r+1}(K)$ *(for* $r \geqslant 1$*).*

p. 16

Proof. Let F' be the filtration on K defined by

$$F'^p(K^n) = \mathrm{Dec}(F)^{p-n}(K^n) = Z_1^{p,n-p}.$$

We trivially have isomorphisms

$$E_r^{p,n-p}(\mathrm{Dec}(K)) = E_{r+1}^{p+n,-p}(K,F') \tag{1.3.4.1}$$

that are compatible with the d_r and with (1.3.1.5). The map u comes from (1.3.4.1) and from the identity map

$$(K,F') \to (K,F).$$

This proves (i), and it remains to show that, for $r \geqslant 2$,

$$E_r^{pq}(K,F') \xrightarrow{\sim} E_r^{pq}(K,F).$$

We have

$$Z_r^{pq}(K,F') = Z_r^{pq}(K,F) \qquad (\text{for } r \geqslant 1)$$
$$Z_r^{pq}(K,F') \cap B_r^{pq}(K,F') = Z_r^{pq}(K,F) \cap B_r^{pq}(K,F) \qquad (\text{for } r \geqslant 2)$$

and we can then apply (1.3.1.2). □

1.3.5. The construction (1.3.3) is not self-dual. The dual construction consists of defining

$$\mathrm{Dec}^\bullet(F)^p K^n = B_1^{p+n-1,-p+1}.$$

We then have morphisms

$$E_0^{p,n-p}(\mathrm{Dec}(K)) \to E_1^{p+n,p}(K) \to E_0^{p,n-p}(\mathrm{Dec}^\bullet(K))$$

and, for $r \geqslant 1$, isomorphisms

$$E_r^{p,n-p}(\mathrm{Dec}(K)) \xrightarrow{\sim} E_{r+1}^{p+n,p}(K) \xrightarrow{\sim} E_r^{p,n-p}(\mathrm{Dec}^\bullet(K)).$$

Recall that a morphism of complexes is said to be a *quasi-isomorphism* if it induces an isomorphism on cohomology.

Definition 1.3.6.

—

i. A morphism $f:(K,F)\to(K',F')$ of filtered complexes with biregular filtrations is a *filtered quasi-isomorphism* if $\mathrm{Gr}_F(f)$ is a quasi-isomorphism, i.e. if the $E_1^{pq}(f)$ are isomorphisms.
ii. A morphism $f:(K,F,W)\to(K,F',W')$ of biregular bifiltered complexes is a *bifiltered quasi-isomorphism* if $\mathrm{Gr}_F\,\mathrm{Gr}_W(f)$ is a quasi-isomorphism.

1.3.7. Let K be a differential complex of objects of $\mathscr{A}$, endowed with two filtrations F and W. Let E_r^{pq} be the spectral sequence defined by W. The filtration F induces on E_r^{pq} various filtrations, which we will compare.

1.3.8. Equation (1.3.1.2) identifies E_r^{pq} with a quotient of a sub-object of K^{p+q}. The E_r^{pq} term is thusly given by endowing with a filtration F_d induced by F, called the *first direct filtration*.

1.3.9. Dually, Equation (1.3.1.3) identifies E_r^{pq} with a sub-object of a quotient of K^{p+q}, whence a new filtration F_{d^*} induced by F, called the *second direct filtration*.

Lemma 1.3.10. *On E_0 and E_1, we have $F_d=F_{d^*}$.* p. 17

Proof. For $r=0,1$, we have $B_r^{pq}\subset Z_r^{pq}$, and we apply (1.1.9). □

1.3.11.

Equation (1.3.1.5) identifies E_{r+1}^{pq} with a quotient of a sub-object of E_r^{pq}. We define the *recurrent filtration* F_r on the E_r^{pq} by the conditions

i. On E_0^{pq}, $F_r=F_d=F_{d^*}$.
ii. On E_{r+1}^{pq}, the recurrent filtration is that induced by the recurrent filtration of E_r^{pq}.

1.3.12. Definitions (1.3.8) and (1.3.9) still make sense for $r=\infty$. If the filtration on K is biregular, then the direct filtrations on E_∞^{pq} coincide with those on $E_r^{pq}=E_\infty^{pq}$ for large enough r, and we define the recurrent filtration on E_∞^{pq} as agreeing with that on E_r^{pq} for large enough r.

The filtrations F and W each induce a filtration on $H^\bullet(K)$, and $E_\infty^{\bullet\bullet}=\mathrm{Gr}_W^\bullet(\mathrm{H}^\bullet(K))$. The filtration F on $\mathrm{H}^\bullet(K)$ then induces on E_∞^{pq} a new filtration.

Proposition 1.3.13.

—

i. For the first direct filtration, the morphisms d_r are compatible with the filtrations. If E_{r+1}^{pq} is considered as a quotient of a sub-object of E_r^{pq}, then the first direct filtration on E_{r+1}^{pq} is finer than the filtration F' induced by the first direct filtration on E_r^{pq}Y we have $F_d(E_{r+1}^{pq})\subset F'(E_{r+1}^{pq})$.
ii. Dually, the morphisms d_r are compatible with the second direct filtration, and the second direct filtration on E_{r+1}^{pq} is less fine than the filtration induced by that of E_r^{pq}.

iii. $F_d(E_r^{pq}) \subset F_r(E_r^{pq}) \subset F_{d^*}(E_r^{pq})$.
iv. *On E_∞^{pq}, the filtration induced by the filtration F of $\mathrm{H}^\bullet(K)$ (1.3.12) is finer than the first direct filtration and less fine than the second.*

Proof. Claim (i) is evident, (ii) is its dual, and (iii) follows by induction. The first claim of (iv) is easy to verify, and the second is its dual. □

1.3.14. We denote by $\mathrm{Dec}(K)$ (resp. $\mathrm{Dec}^\bullet(K)$) the complex K endowed with the filtrations $\mathrm{Dec}(W)$ and F (resp. $\mathrm{Dec}^\bullet(W)$ and F).

It is clear by (1.3.4.1) that the isomorphism (1.3.4) sends the first direct filtration on $E_r(\mathrm{Dec}(K))$ to the second direct filtration on $E_{r+1}(K)$ (for $r \geqslant 1$). The dual isomorphism (1.3.5) sends the second direct filtration on $E_r(\mathrm{Dec}^\bullet(K))$ to the second direct filtration on $E_{r+1}(K)$.

Lemma 1.3.15.

If the filtration F is biregular, and if, on the $\mathrm{Gr}_W^p(K)$, the morphisms d are strictly compatible with the filtration induced by F, then

i. *The morphism (1.3.3.2) of graded complexes filtered by F*
$$u\colon \mathrm{Gr}_{\mathrm{Dec}(W)}(K) \to E_1(K,W)$$
is a filtered quasi-isomorphism.
ii. *Dually, the morphism (1.3.5)*
$$u\colon E_1(K,W) \to \mathrm{Gr}_{\mathrm{Dec}^\bullet(W)}(K)$$
is a filtered quasi-isomorphism.

Proof. It suffices, by duality, to prove (i). *p. 18*

By (1.3.3) and (1.3.4), the complex $E_1(K,W)$ filtered by F is a quotient of the filtered complex $\mathrm{Gr}_{\mathrm{Dec}(W)}(K)$. Let U be the filtered complex given by the kernel, which is acyclic by (1.3.4), (ii). The long exact sequence in cohomology associated to the exact sequence of complexes
$$0 \to \mathrm{Gr}_F(U) \to \mathrm{Gr}_F(\mathrm{Gr}_{\mathrm{Dec}(W)}(K)) \to \mathrm{Gr}_F(E_1(K,W)) \to 0$$
shows that u is a filtered quasi-isomorphism if and only if $\mathrm{Gr}_F(U)$ is an acyclic complex. By (1.3.2), and since U is acyclic, this reduces to asking that the differentials of U be strictly compatible with the filtration F. From (1.3.3.1) we obtain that U is the sum over p of the complexes
$$(U^p)^n = B_1^{p+n,-p}/Z_1^{p+1+n,-p-1}$$
endowed with the filtration induced by F.

Each differential d of each of the complexes U^p fits into a commutative diagram of filtered objects of the following type, where, for simplicity, we have omitted the total or complementary degree: **!!TO-DO: diagram!!** By hypothesis, the morphism (1) is strict. Since the square (2) is exactly the canonical decomposition of (1), the arrow (3) is a filtered isomorphism. The arrows of the trapezium (4) are isomorphisms; they are thus filtered isomorphisms, since (3) is a filtered isomorphism. The fact that (5) is a filtered isomorphism implies that d is strict. This proves the lemma. □

Theorem 1.3.16. *Let K be a complex endowed with two filtrations, W and F, with the filtration F biregular. Let $r_0 \geqslant 0$ be an integer, and suppose that, for $0 \leqslant r < r_0$, the differentials of the graded complex $E_r(K,W)$ are strictly compatible with the filtration F. Then, for $r \leqslant r_0+1$, $F_d = F_r = F_{d^*}$ on E_r^{pq}.*

Proof. We will prove the theorem by induction on r_0. For $r_0 = 0$, the hypothesis is empty, and we apply (1.3.10) and (1.3.13), (iii). For $r_0 \geqslant 1$, by the inductive hypothesis, we have $F_d = F_r = F_{d^*}$ on E_r^{pq} for $r \leqslant r_0$. *p. 19*

By (1.3.15), the morphism $u\colon E_0(\mathrm{Dec}(K)) \to E_1(K)$ is a filtered quasi-isomorphism. It thus induces a filtered isomorphism from $\mathrm{H}^\bullet(\mathrm{Dec}(K))$ to $\mathrm{H}^\bullet(E_1(K))$:

$$u\colon (E_1(\mathrm{Dec}(K)),F_r) \xrightarrow{\sim} (E_2(K),F_r).$$

Step-by-step, we thus deduce that the canonical isomorphism from $E_s(\mathrm{Dec}(K))$ to E_{s+1} (for $s \geqslant 1$) is a filtered isomorphism for the recurrent filtration.

On $E_1(\mathrm{Dec}(K))$, $F_r = F_d$ (1.3.10), and we already know (1.3.14) that u' is a filtered isomorphism

$$u'\colon (E_1(\mathrm{Dec}(K)),F_d) \xrightarrow{\sim} (E_2(K),F_d).$$

On $E_2(K)$, we thus have $F_d = F_r$. This, combined with the dual result, proves the theorem for $r_0 = 1$.

Suppose that $r_0 \geqslant 2$. Then the arrows d_1 of $E_1(K)$ are strictly compatible with the filtrations, and thus so too are the arrows d_0 of $E_0(\mathrm{Dec}(K))$ (indeed, u induces an isomorphism of spectral sequences, and we apply criterion (1.3.2)).

For $0 < s < r_0 - 1$, the isomorphism $(E_s(\mathrm{Dec}(K)),F_r) \cong (E_{s+1}(K),F_r)$ shows that the d_s are strictly compatible with the recurrent filtrations.

By the induction hypothesis, we thus have $F_d = F_r$ on $E_s(\mathrm{Dec}(K))$ for $s \leqslant s_0$. The isomorphism $(E_s(\mathrm{Dec}(K)),F_d) \cong (E_{s+1}(K),F_d)$ (1.3.13) then shows that $F_d = F_r$ on $E_r(K)$ for $r \leqslant r_0 + 1$. This, combined with the dual result, proves the theorem. ☐

Corollary 1.3.17. *Under the general hypotheses of (1.3.16), suppose that, for all r, the differentials d_r are strictly compatible with the recurrent filtrations on the E_r. Then, on E_∞, the filtrations F_d, F_r, and F_{d^*} agree, and coincide with the filtration induced by the filtration F of $\mathrm{H}^\bullet(K)$.*

Proof. This follows immediately from (1.3.16) and (1.3.13), (iv). ☐

1.4 Hypercohomology of filtered complexes

In this section, we recall some standard constructions in hypercohomology. We do not use the language of derived categories, which would be more natural here.

Throughout this entire section, by "complex" we mean "bounded-below complex."

1.4.1. Let T be a left-exact functor from an abelian category $\mathscr{A}$ to an abelian category $\mathscr{B}$. Suppose that every object of $\mathscr{A}$ injects into an injective object; the derived functors

$\mathrm{R}^i T\colon \mathscr{A} \to \mathscr{B}$ are then defined. An object A of $\mathscr{A}$ is said to be *acyclic* for T if $\mathrm{R}^i T(A) = 0$ for $i > 0$.

1.4.2. Let (A,F) be a filtered object with finite filtration, and TF the filtration of TA by its sub-objects $TF^p(A)$ (these are sub-objects since T is left exact). If $\mathrm{Gr}_F(A)$ is T-acyclic, then the $F^p(A)$ are T-acyclic as successive extensions of T-acyclic objects. The image under T of the sequence

$$0 \to F^{p+1}(A) \to F^p(A) \to \mathrm{Gr}^p(A) \to 0$$

is thus exact, and p. 20

$$\mathrm{Gr}_{FT}\, TA \xrightarrow{\sim} T\,\mathrm{Gr}_F\, A. \tag{1.4.2.1}$$

1.4.3. Let A be an object endowed with finite filtrations F and W such that $\mathrm{Gr}_F\,\mathrm{Gr}_W\, A$ are T-acyclic. The objects $\mathrm{Gr}_F\, A$ and $\mathrm{Gr}_W\, A$ are then T-acyclic, as well as the $F^q(A) \cap W^p(A)$. The sequences

$$0 \to T(F^q \cap W^{p+1}) \to T(F^q \cap W^p) \to T((F^q \cap W^p)/(F^q \cap W^{p+1})) \to 0$$

are thus exact, and $T(F^q(\mathrm{Gr}^p_W(A)))$ is the image in $T(\mathrm{Gr}^p_W(A))$ of $T(F^p \cap W^q)$. The diagram

$$\begin{array}{ccccc}
T(F^q \cap W^p) & \longrightarrow & T(F^q\,\mathrm{Gr}^p_W A) & \longrightarrow & T\,\mathrm{Gr}^p_W A \\
{\scriptstyle\cong}\big\downarrow & & & & \big\downarrow{\scriptstyle\cong} \\
TF^q \cap TW^p & = & TF^q \cap TW^p & \longrightarrow & \mathrm{Gr}^p_{TW}\, TA
\end{array}$$

then shows that the isomorphism (1.4.2.1) relative to W sends the filtration $\mathrm{Gr}_{TW}(TF)$ to the filtration $T(\mathrm{Gr}_W(F))$.

1.4.4. Let K be a complex of objects of $\mathscr{A}$. The *hypercohomology objects* $\mathrm{R}^i T(K)$ are calculated as follows:

a. We choose a quasi-isomorphism $i\colon K \to K$ such that the components of K' are acyclic for T. For example, we can take K' to be the simple complex associated to an injective Cartan–Eilenberg resolution of K.
b. We set
$$\mathrm{R}^i T(K) = \mathrm{H}^i(T(K')).$$

We can show that $\mathrm{R}^i T(K)$ does not depend on the choice of K', but depends functorially on K, and that a quasi-isomorphism $f\colon K_1 \to K_2$ induces *isomorphisms*

$$\mathrm{R}^i T(f)\colon\ \mathrm{R}^i T(K_1) \to \mathrm{R}^i T(K_2).$$

1.4.5. Let F be a biregular filtration of K. A *T-acyclic filtered resolution* of K is a filtered quasi-isomorphism $i\colon K \to K'$ from K to a filtered biregular complex such that the

$\mathrm{Gr}^p(K'^n)$ are acyclic for T. If K' is such a resolution, then the K'^n are acyclic for T, and the filtered complex (cf. (1.4.2)) $T(K')$ defines a spectral sequence

$$E_1^{pq} = \mathrm{R}^{p+q}\,T(\mathrm{Gr}^p(K)) \Rightarrow \mathrm{R}^{p+q}\,T(K).$$

This is independent of the choice of K'. We call this the *hypercohomology spectral sequence of the filtered complex* K. It depends functorially on K, and a filtered quasi-isomorphism induces an isomorphism of spectral sequences. *p. 21*

The differentials d_1 of this spectral sequence are the connection morphisms defined by the short exact sequences

$$0 \to \mathrm{Gr}^{p+1}K \to F^pK/F^{p+2}K \to \mathrm{Gr}^p K \to 0.$$

1.4.6. Let K be a complex.

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