

# Quasi-coherent sheaves

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## Translator's note.

*This text is one of a series\* of translations of various papers into English. The translator takes full responsibility for any errors introduced in the passage from one language to another, and claims no rights to any of the mathematical content herein.*

*What follows is a translation of the French seminar talk:*

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We assume prior knowledge of the definitions and elementary properties of sheaves of modules on a topological space, i.e. [2, chapitre I, §1; chapitre II, §§1–2]. We define a presheaf  $\mathcal{P}$  on a base  $\mathcal{B}$  of open subsets of a topological space  $X$ , with values in a category  $\mathcal{C}$ , to be the following data:

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- (a) for every open subset  $U$  in  $\mathcal{B}$ , an object  $\mathcal{P}(U)$  of  $\mathcal{C}$ , that we may also denote by  $\Gamma(U, \mathcal{P})$ ;
- (b) for every pair  $(U, V)$  of open subsets  $U$  in  $\mathcal{B}$  such that  $U \subset V$ , a morphism  $\rho_{UV}: \mathcal{P}(V) \rightarrow \mathcal{P}(U)$ . The morphism  $\rho_{UV}$  will be called the restriction of  $V$  to  $U$ . We further suppose that, if  $U \subset V \subset W$  are open subsets in  $\mathcal{B}$ , then  $\rho_{UV} \circ \rho_{VW} = \rho_{UW}$ .

The construction of the sheaf  $\widetilde{\mathcal{P}}$  associated to a presheaf  $\mathcal{P}$  can be easily generalised to the case of presheaves on a base of open subsets. If  $\mathcal{U} = (U_i)_{i \in I}$  is a cover of the open subset  $U \in \mathcal{B}$ , where  $U_i \in \mathcal{B}$ , and  $U_i \cap U_j \in \mathcal{B}$ , and if  $\mathcal{P}$  is a presheaf of rings (resp. of

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modules) on the base  $\mathcal{B}$ , then we write  $H^0(\mathcal{U}, \mathcal{P})$  to mean the subring (resp. submodule) of  $\prod_{i \in I} \mathcal{P}(U_i)$  given by the  $(X_i)_{i \in I}$  such that  $\rho_{U_i \cap U_j, U_i}(X_i) = \rho_{U_i \cap U_j, U_j}(X_j)$  for every pair  $i, j \in I$ . We then have canonical maps:

$$\mathcal{P}(U) \rightarrow H^0(\mathcal{U}, \mathcal{P}) \rightarrow \widetilde{\mathcal{P}}(U);$$

if these maps are injective for every cover  $\mathcal{U}$  satisfying the conditions above, then  $\widetilde{\mathcal{P}}(U)$  is the union of the  $H^0(\mathcal{U}, \mathcal{P})$ .

## 1 Preliminaries on localisation

Let  $A$  be a unital commutative ring. A submonoid  $S$  of the multiplicative monoid of  $A$  (i.e. a non-empty subset of  $A$  such that, if it contains  $s$  and  $t$ , then it contains  $s \cdot t$ ) is said to be *complete* if it satisfies the following condition: if  $s \cdot t \in S$ , then  $s \in S$  and  $t \in S$ . To every multiplicative submonoid  $S$ , we associate a complete monoid  $\widetilde{S}$  in the following way:  $s \in \widetilde{S}$  if and only if there exists some  $t$  in  $A$  such that  $s \cdot t \in S$ . The prime ideals that meet  $S$  also meet  $\widetilde{S}$ , and vice versa. Furthermore, the complement of  $\widetilde{S}$  in  $A$  is a union of prime ideals (the ideals that are maximal amongst those that do not meet  $\widetilde{S}$  are prime).

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If now  $M$  denotes a unital  $A$ -module, and  $S$  a multiplicative submonoid of  $A$ , then we denote by  $M_S$  the following abelian group:

The set  $M_S$  is the quotient of  $M \times S$  by the equivalence relation

$$(m, s) = (n, t) \iff \exists r \in S \text{ such that } r(mt - ns) = 0.$$

The addition in  $M \times S$  is defined by  $(m, s) + (n, t) = (mt + ns, st)$ ; our equivalence relation is compatible with the addition, whence we get the structure of an abelian group on  $M_S$ . We denote by  $m/s$  the class of  $(m, s)$  in  $M_S$ .

We have a bilinear map  $A_S \otimes_{\mathbb{Z}} M_S \rightarrow M_S$  defined by passing to quotients from the map  $((a, s), (m, t)) \mapsto (am, st)$ . Taking  $M = A$ , we see that  $A_S$  is a ring and, more generally,  $M_S$  is an  $A_S$ -module. The map  $\psi: m \mapsto m/1$  from  $M$  to  $M_S$  is compatible with the ring homomorphism  $\varphi: a \mapsto a/1$  from  $A$  to  $A_S$  (i.e. it is a homomorphism of abelian groups such that  $\psi(a \cdot m) = \varphi(a) \cdot \psi(m)$ ).

We can furthermore easily prove the following claims: the correspondence  $M \mapsto M_S$  is functorial (in the evident way); the functor  $M \mapsto M_S$  is exact (i.e. if  $0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0$  is an exact sequence of  $A$ -modules, then  $0 \rightarrow M_S \rightarrow M'_S \rightarrow M''_S \rightarrow 0$  is an exact sequence of  $A$ -modules) and commutes with inductive limits (i.e. if  $(M_i)_{i \in I}$  is an inductive system of  $A$ -modules, and  $L$  an inductive limit of this system, then the  $(M_i)_S$  form an inductive system of  $A_S$ -modules, and the homomorphisms  $(M_i)_S \rightarrow L_S$  induced by the  $M_i \rightarrow L$  define  $L_S$  as an inductive limit of the system  $((M_i)_S)$ ; the canonical map  $M \otimes_A A_S \rightarrow M_S$  is bijective.

If  $S$  and  $T$  are submonoids, then the abelian groups  $(M_S)_T$ ,  $(M_T)_S$ , and  $M_{S \cdot T}$  are all canonically isomorphic to one another, where  $S \cdot T$  denotes the monoid given by products  $s \cdot t$  with  $s \in S$  and  $t \in T$ . Identifying the rings  $(A_S)_T$ ,  $(A_T)_S$ , and  $A_{S \cdot T}$ , which are themselves all canonically isomorphic to one another, the isomorphisms between  $(M_S)_T$ ,  $(M_T)_S$ , and  $M_{S \cdot T}$  are isomorphisms of modules.

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If  $S$  is contained in a submonoid  $S'$  of the multiplicative monoid of  $A$ , then there is a canonical map from  $M_S$  to  $M_{S'}$  that sends each element  $m_s$  of  $M_S$  (where  $m \in M$  and  $s \in S$ ) to the element of  $M_{S'}$  denoted by the same symbol. The map  $A_S \rightarrow A_{S'}$  is a ring

homomorphism; the map  $M_S \rightarrow M_{S'}$  is that which corresponds to the map  $M \otimes_A A_S \rightarrow M \otimes_A A_{S'}$ , induced by  $A_S \rightarrow A_{S'}$ . If  $S' = \tilde{S}$  is the smallest complete monoid containing  $S$ , then the map  $M_S \rightarrow M_{\tilde{S}}$  is bijective. Finally, if  $S$  is the union of an increasingly-ordered filtered family of monoids  $S_i$ , then  $M_S$  can be identified with  $\varinjlim M_{S_i}$ .

## 2 The prime spectrum of a commutative ring

Let  $A$  be a unital commutative ring, and  $V(A)$  the set of prime ideals of  $A$ . If  $\mathfrak{a}$  is an ideal of  $A$ , then we denote by  $W(\mathfrak{a})$  the set of prime ideals that contain  $\mathfrak{a}$ , and  $U(\mathfrak{a}) = V(A) \setminus W(\mathfrak{a})$ .

Then

$$\begin{aligned} W(\mathfrak{a}) &\subset V(A), & U(\mathfrak{a}) &\subset V(A), \\ W(\sum \mathfrak{a}_i) &= \bigcap_i W(\mathfrak{a}_i), & W(\mathfrak{a} \cap \mathfrak{b}) &= W(\mathfrak{a} \cdot \mathfrak{b}) = W(\mathfrak{a}) \cup W(\mathfrak{b}), \\ U(\sum \mathfrak{a}_i) &= \bigcup_i U(\mathfrak{a}_i), & U(\mathfrak{a} \cap \mathfrak{b}) &= U(\mathfrak{a} \cdot \mathfrak{b}) = U(\mathfrak{a}) \cap U(\mathfrak{b}). \end{aligned}$$

The set  $U(\mathfrak{a})$  increases with  $\mathfrak{a}$  and  $W(\mathfrak{a})$  decreases when  $\mathfrak{a}$  increases. Finally,  $W(\mathfrak{a}) = W(\mathfrak{b})$  if and only if every element of  $\mathfrak{a}$  has a power in  $\mathfrak{b}$ , and vice versa: indeed, to say that  $W(\mathfrak{a}) \subset W(\mathfrak{b})$  is to say that every prime ideal that contains  $\mathfrak{a}$  also contains  $\mathfrak{b}$ , i.e. that  $\mathfrak{b}$  is contained in the intersection of the prime ideals containing  $\mathfrak{a}$ , and it is classical that this latter intersection consists of the elements that have a power in  $\mathfrak{a}$ .

The sets  $U(\mathfrak{a})$  are the open subsets of a topology on  $V(A)$ , and, endowed with this topology,  $V(A)$  is called the *prime spectrum* of  $A$ .

If the ideal  $\mathfrak{a}$  is generated by the  $(f_i)$ , then  $U(\mathfrak{a})$  is the union of the  $U((f_i)) = U_{f_i}$ ; since  $U_f \cap U_g = U_{fg}$ , the  $U_f$  thus form a base of open subsets when  $f$  runs over the elements of  $A$ . Every open subset of the form  $U_f$  is said to be *special*. Every special open subset is *quasi-compact*: indeed, if  $U_f$  is the union of some  $U(\mathfrak{a}_i)$ , then  $U_f = \bigcup_i U(\mathfrak{a}_i) = U(\sum \mathfrak{a}_i)$ , and a power  $f^n$  of  $f$  belongs to  $\sum_i \mathfrak{a}_i$ , and thus to the sum of a finite number of the  $\mathfrak{a}_i$ .

In particular,  $V = U_1$  is quasi-compact. Finally, if the ring  $A$  is *Noetherian*, then every increasing sequence of ideals stabilises, and the same is true for every increasing sequence of open subsets. The prime spectrum is thus a *Zariski topological space*.

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## 3 Quasi-coherent sheaves on $V(A)$

Now let  $M$  be a unital module over the ring  $A$ . We are going to define a *presheaf*  $\mathfrak{M}^*$  on the base of special open subsets of  $V(A)$ . For this, we will associate, to each  $f \in A$ , the multiplicatively-stable system  $S_f$  given by the complement in  $A$  of the union of the prime ideals of  $U_f$ . The system  $S_f$  is the complete multiplicatively-stable system generated by  $f$ . We then have the equivalence:

$$S_f \subset S_g \mapsto U_f \supset U_g.$$

With this in mind, we will define the presheaf  $\mathfrak{M}^*$  by the formulas  $\mathfrak{M}^*(U_f) = M_{S_f}$ , which we will also denote by  $\mathfrak{M}_f$ ; if  $U_f \supset U_g$ , then the restriction map  $\rho: \mathfrak{M}^*(U_f) \rightarrow \mathfrak{M}^*(U_g)$  is the canonical map from  $\mathfrak{M}_{S_f}$  to  $\mathfrak{M}_{S_g}$ . These definitions clearly do not depend

on the elements  $f$  and  $g$  that define  $U_f$  and  $U_g$ ; furthermore, the axioms of a presheaf are satisfied, thanks to the properties of localisation.

We will denote by  $\mathfrak{M}$  the sheaf associated to the presheaf  $\mathfrak{M}^*$ . Since the  $A_f$  are rings, and the  $M_f$  are  $A_f$ -modules,  $\mathfrak{A}^*$  is a presheaf of rings,  $\mathfrak{M}^*$  a presheaf of  $\mathfrak{A}^*$ -modules,  $\mathfrak{A}$  a sheaf of rings, and  $\mathfrak{M}$  a sheaf of  $\mathfrak{A}$ -modules. Furthermore, the functors that send  $M$  to  $\mathfrak{M}^*$  and  $\mathfrak{M}$  are exact, since the functor that sends  $M$  to the  $M_f$  is exact. We define an *algebraic sheaf* on  $V(A)$  to be any sheaf of modules over the sheaf of rings  $\mathfrak{A}$ . We define a *quasi-coherent sheaf* on  $V(A)$  to be any sheaf isomorphic to a sheaf of the form  $\mathfrak{A}(M)$  for some  $M$ . We will first be interested in the sections of such a sheaf over a special open subset  $U_f$ .

For this, we note that the prime ideals of  $A_f$  are the images under the map  $A \rightarrow A_f$  of the prime ideals of  $A$  that do not contain  $f$ . Furthermore, the canonical application thus defined from the spectrum  $V(A_f)$  of  $A_f$  to  $V(A)$  is a homomorphism from  $V(A_f)$  to  $U_f$ . Finally, the sheaf on  $V(A_f)$  associated to the module  $M_f$  can be identified with the restriction of  $\mathfrak{M}$  to  $U_f$ .

More generally, if  $S$  is a multiplicatively-stable system (that we can assume to be complete) of  $A$ , then let  $E_S$  be the topological subspace of  $V(A)$  consisting of the prime ideals that do not meet  $S$ . The canonical map  $A \rightarrow A_S$  induces a homomorphism from  $V(A_S)$  to  $E_S$ ; if  $S$  is generated by the  $f_i$ , then  $E_S$  is the intersection of the  $U_{f_i}$ , and, conversely, every intersection of special open subsets of  $V(A)$  is of the form  $E_S$ . We will see that the restriction of  $\mathfrak{M}$  to the subspace  $E_S$  can be identified with the sheaf  $\mathfrak{M}_S$  associated to the  $A_S$ -module  $M_S$ .

For every prime ideal  $\mathfrak{p}$ , we denote by  $A_{\mathfrak{p}}$  and  $M_{\mathfrak{p}}$  the ring and the module obtained by localisation of the multiplicative system  $S = A \setminus \mathfrak{p}$ .

**Theorem 1.**

- (a) If  $\mathfrak{p}$  is a point of  $V(A)$ , then the localised ring  $\mathfrak{A}_{\mathfrak{p}}$  (the fibre of the sheaf  $\mathfrak{A}$  over  $\mathfrak{p}$ ) is canonically isomorphic to  $A_{\mathfrak{p}}$ ; Identifying  $\mathfrak{A}_{\mathfrak{p}}$  with  $A_{\mathfrak{p}}$ , the localised module  $\mathfrak{M}_{\mathfrak{p}}$  of the sheaf  $\mathfrak{M}$  at  $\mathfrak{p}$  is canonically isomorphic to  $M_{\mathfrak{p}}$ , and thus to  $M \otimes_A A_{\mathfrak{p}}$ .
- (b) The canonical map from  $M_f = \mathfrak{M}^*(U_f)$  to  $\mathfrak{M}(U_f)$  is an isomorphism.

*Proof.* The first claim of (a) follows from the fact that

$$\mathfrak{A}_{\mathfrak{p}} = \varinjlim_{U_f \ni \mathfrak{p}} \mathfrak{A}^*(U_f) = \varinjlim_{f \notin \mathfrak{p}} A_f = A_{\mathfrak{p}};$$

the second claim of (a) can be proven in the same way.

We now show that the map  $M_f \rightarrow \mathfrak{M}(U_f)$  is *injective*: Since  $U_f = V(A_f)$ , we can always assume that  $f = 1$  and  $U_f = V(A)$ . The kernel of the map  $M \rightarrow \mathfrak{M}(V)$  then consists of the  $m$  such that  $m \otimes_A 1_{A_{\mathfrak{p}}} = 0$  for every prime ideal  $\mathfrak{p}$ , i.e. the  $m$  whose annihilator is not contained in any prime ideal: the kernel is thus zero.

Similarly, the map  $M_f \rightarrow \mathfrak{M}(U_f)$  is *surjective*: it suffices to prove this in the case where  $f = 1$  and  $U_f = V$ . By the above,  $\mathfrak{M}(V) = \Gamma(\mathfrak{M})$  is the union of the  $H^0(\mathfrak{U}, \mathfrak{M}^*)$  where  $\mathfrak{U}$  runs over the finite covers of  $V$  by special open subsets; it thus suffices to prove that the map  $M \rightarrow H^0(\mathfrak{U}, \mathfrak{M})$  is surjective for every finite cover by special open subsets.

| p. 1-05

For this, let  $(f_i)_{i=1,\dots,p}$  be elements of  $A$  such that  $V = \bigcup U_{f_i}$ , and let  $X_i$  be the elements of  $M_{f_i}$  such that  $X_i$  and  $X_j$  have the same image in  $M_{f_i \cdot f_j}$ ; even if we replace  $f_i$  by one of its powers, we can still suppose that  $X_i$  is of the form  $m_i/f_i$ , where  $m_i \in M$ . Since  $X_i$  and  $X_j$  have the same image in  $M_{f_i \cdot f_j}$ ,

$$\frac{m_i f_j - m_j f_i}{f_i f_j} = 0 \in M_{f_i \cdot f_j}.$$

In other words,

$$(f_i \cdot f_j)^r (m_i f_j - m_j f_i) = 0 \in M \text{ for } r \text{ large enough.}$$

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Also, the  $U_{f_i} = U_{f_i^{r+1}}$  cover  $V$ , i.e. the  $f_i^{r+1}$  generate  $A$ , i.e. we have a relation of the form  $1 = \sum_{i=1}^p a_i f_i^{r+1}$ , where  $a_i \in A$ . It follows that, if  $m = \sum_i m_i a_i f_i^{r+1}$ , then

$$f_j^{r+1} = \sum_i a_i m_i f_i^r f_j^{r+1} = \sum_i a_i m_j f_j^r f_i^{r+1} = m_j f_j^r$$

and, in  $M_{f_j}$ , we have the equality  $X_j = m_j/f_j = m/1$  for all  $j$ .  $\square$

**Corollary 1.** *If  $S$  is a multiplicatively-stable subset of  $A$ , then the restriction of the sheaf  $\mathfrak{M}$  to  $E_S$  can be identified with the sheaf  $\mathfrak{M}_S$  on  $V(A_S)$  associated to  $M_S$ .*

*Proof.* Let  $f$  be an element of  $A$ . The open subset  $E_S \cap U_f$  of  $E_S$  can then be identified with  $E_{S \cdot S_f}$ , and, consequently,

$$\mathfrak{M}_S(E_S \cap U_f) = M_{S \cdot S_f} = (M_S)_f = \varinjlim_{U_g \supset E_S} M_{f \cdot g}.$$

The natural maps  $M_{f \cdot g} = \mathfrak{M}(U_{fg}) \rightarrow \mathfrak{M}(E_S \cap U_f)$  thus extend to give a map

$$\mathfrak{M}_S(E_S \cap U_f) \rightarrow \mathfrak{M}(E_S \cap U_f)$$

and induce a morphism of sheaves:

$$\mathfrak{M}_S \rightarrow \mathfrak{M}|_{E_S}.$$

But if  $p \in E_S$ , then the fibre of  $\mathfrak{M}_S$  over  $p$  is exactly  $(M_S)_p = M_p$ , and the above map induces an isomorphism over each  $p$ : the map is thus an isomorphism of sheaves.  $\square$

| p. 1-07

**Corollary 2.**  *$M_S$  is the module of sections of  $\mathfrak{M}$  over  $E_S$ .*

**Corollary 3.** *If  $M$  and  $N$  are  $A$ -modules, then the map*

$$\varphi: \text{Hom}(M, N) \rightarrow \text{Hom}(\mathfrak{M}, \mathfrak{N})$$

*defined by the functor  $M \mapsto \mathfrak{M}$  is bijective.*

*Proof.* We will exhibit an inverse map  $\psi$ . For this, let  $\Gamma$  be the functor that sends any sheaf of  $\mathfrak{A}$ -modules to its module of sections over  $V(A)$ . The functor  $\Gamma$  defines a map from  $\text{Hom}(\mathfrak{M}, \mathfrak{N})$  to  $\text{Hom}(\Gamma(\mathfrak{M}), \Gamma(\mathfrak{N}))$ , and thus, since this latter group is isomorphic to  $\text{Hom}(M, N)$  by an explicit isomorphism described above, a map  $\psi$  from  $\text{Hom}(\mathfrak{M}, \mathfrak{N})$  to  $\text{Hom}(M, N)$ . Then  $\varphi$  and  $\psi$  are inverse to one another.  $\square$

It follows from [this corollary](#), along with the exactness of the functor  $M \mapsto \mathfrak{M}$ , that, if  $u: \mathcal{F} \rightarrow \mathcal{G}$  is a morphism of quasi-coherent sheaves, then  $\text{Ker } u$  and  $\text{Coker } u$  are quasi-coherent sheaves.

**Theorem 2.** *If  $\mathcal{F}$  is an algebraic sheaf on  $V(A)$ , then the following are equivalent:*

(a)  $\mathcal{F}$  is quasi-coherent.

(b) For every special open subset  $U_f$ , the natural map

$$\Gamma(V, \mathcal{F}) \otimes_A A_f \rightarrow \Gamma(U_f, \mathcal{F})$$

is bijective.

(c) For every point  $\mathfrak{p}$  of  $V$ , the natural map

$$\Gamma(V, \mathcal{F}) \otimes_A A_{\mathfrak{p}} \rightarrow \mathcal{F}_{\mathfrak{p}}$$

is bijective.

(d)  $\mathcal{F}$  is locally isomorphic to a quasi-coherent sheaf, i.e. every point  $\mathfrak{p}$  of  $V$  has a special neighbourhood  $U_f$  such that  $\mathcal{F}|_{U_f}$  is a coherent sheaf.

*Proof.* We have already seen that (a)  $\implies$  (b), (a)  $\implies$  (c), and (a)  $\implies$  (d).

We will first show that (b)  $\implies$  (a): Let  $M = \Gamma(V, \mathcal{F})$ . Then, for every affine open subset  $U_f$  we have restrictions:  $M \rightarrow \Gamma(U_f, \mathcal{F})$ , and since  $\Gamma(U_f, \mathcal{F})$  is an  $A_f$ -module, it follows that we have a map:

$$M_f = M \otimes_A A_f \rightarrow \Gamma(U_f, \mathcal{F}).$$

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These maps induce a morphism of sheaves:  $\mathfrak{M} \rightarrow \mathcal{F}$ .

This morphism will be bijective if the induced maps  $\mathfrak{M}(U_f) \rightarrow \mathcal{F}(U_f)$  are bijective (and so (b)  $\implies$  (a)) or if the induced maps  $\mathfrak{M}_{\mathfrak{p}} \rightarrow \mathcal{F}_{\mathfrak{p}}$  are bijective (and so (c)  $\implies$  (a)).

It remains only to show that (d)  $\implies$  (a). The proof that follows is due to Grothendieck.

We note first of all that the map  $A \rightarrow A_f$  endows every  $A_f$ -module with an  $A$ -module structure, and thus associates, to every quasi-coherent sheaf  $\mathcal{F}$  on  $U_f$ , a quasi-coherent sheaf  $\overline{\mathcal{F}}$  on  $V$  (we will return later to this operation). Furthermore, if  $U_g$  is an affine open subset of  $V$ , then:

$$\Gamma(U_g, \overline{\mathcal{F}}) = \Gamma(V, \overline{\mathcal{F}})_g = \Gamma(U_f, \mathcal{F})_g = \Gamma(U_f \cap U_g, \mathcal{F}).$$

So take some algebraic sheaf  $\mathcal{G}$  on  $V$  such that there exists a cover of  $V$  by special open subsets  $U_{f_i}$  with the condition that  $\mathcal{G}|_{U_{f_i}}$  is quasi-coherent. We will now show that  $\mathcal{G}$  is quasi-coherent.

Since  $\mathcal{G}$  is a sheaf, we have, for all  $U_g$ , an exact sequence of the form:

$$0 \rightarrow \Gamma(U_g, \mathcal{G}) \rightarrow \prod_i \Gamma(U_g \cap U_i, \mathcal{G}) \rightarrow \prod_{i < j} \Gamma(U_g \cap U_i \cap U_j, \mathcal{G})$$

or even

$$0 \rightarrow \Gamma(U_g, \mathcal{G}) \rightarrow \prod_i \Gamma(U_g, \overline{\mathcal{G}|_{U_i}}) \rightarrow \prod_{i \leq j} \Gamma(U_g, \overline{\mathcal{G}|_{U_i \cap U_j}}).$$

The two latter terms of the exact sequence are the sections over  $U_g$  of the sheaves associated to the modules

$$M = \prod_i \Gamma(U_i, \mathcal{G}|_{U_i}) \text{ and } N = \prod_{i < j} \Gamma(U_i \cap U_j, \mathcal{G}).$$

Letting  $g$  vary, we obtain, by taking the limit, an “exact sequence of sheaves”:  $0 \rightarrow \mathcal{G} \rightarrow \mathfrak{M} \rightarrow \mathfrak{N}$ . The sheaf  $\mathcal{G}$  is thus the kernel of a morphism of quasi-coherent sheaves:  $\mathcal{G}$  is quasi-coherent.  $\square$

## 4 Coherent sheaves on $V(A)$

From now on we will suppose that  $A$  is a Noetherian ring. Then it is well known that the category of Noetherian  $A$ -modules agrees with that of  $A$ -modules of finite type. We define a *coherent sheaf on  $V(A)$*  to be any sheaf  $\mathcal{F}$  that is isomorphic to a sheaf of the type  $\mathfrak{M}$ , where  $M$  is an  $A$ -module of finite type. The sheaf  $\mathcal{F}$  is thus quasi-coherent, and it is a sheaf of finite type (a sheaf  $\mathcal{F}$  of  $\mathfrak{A}$ -modules on a topological space  $X$  is said to be of *finite type* if every point of  $X$  admits a neighbourhood  $U$  along with a surjection  $\mathfrak{A}^p|_U \rightarrow \mathcal{F}|_U \rightarrow 0$ , where  $\mathfrak{A}^p$  is sheaf given by the direct sum of  $p$  sheaves, each isomorphic to  $\mathfrak{A}$ ). Such a surjection induces a map

$$\Gamma(U, \mathfrak{A}^p) = \Gamma(U, \mathfrak{A})^p \rightarrow \Gamma(U, \mathcal{F})$$

and defines  $p$  surjections from  $\mathcal{F}$  to  $U$ , i.e. the images of the basis vectors of  $\Gamma(U, \mathfrak{A})^p$ . Conversely, the data of  $p$  sections of  $\mathcal{F}$  over  $U$  defines a map  $\mathfrak{A}^p|_U \rightarrow \mathcal{F}|_U$ .

**Theorem 3.** *The following are equivalent (if  $A$  is Noetherian and if  $\mathcal{F}$  is an algebraic sheaf on  $V(A)$ ):*

- (a)  $\mathcal{F}$  is a coherent algebraic sheaf on  $V(A)$ .
- (b)  $\mathcal{F}$  is quasi-coherent and of finite type.
- (c)  $\mathcal{F}$  is of finite type, and, for every open subset  $U$  and every morphism  $\varphi: \mathcal{G} \rightarrow \mathcal{F}|_U$ , where  $\mathcal{G}$  is a sheaf of finite type on  $U$ , the kernel  $\text{Ker } \varphi$  is of finite type on  $U$ .

*Proof.* We already know that (a)  $\implies$  (b). Conversely, if  $\mathcal{F}$  is quasi-coherent and of finite type, then there exists a cover  $(U_i)_{i \in I}$  of  $V(A)$  by special open subsets such that  $\Gamma(U_{f_i}, \mathcal{F})$  is an  $A_{f_i}$ -module of finite type. Since  $V(A)$  is quasi-compact, we can always assume that this cover is finite, and that  $\mathcal{F}$  is of the form  $\mathfrak{M}$ : we then need to prove that  $M$  is an  $A$ -module of finite type. But, for all  $i$ , there exists a finite number of  $m_{i_k} \in M$  that generate  $M_{f_i}$ , and, if  $N$  denotes the submodule generated by all the  $m_{i_k}$ , then  $N$  is an  $A$ -module of finite type, and clearly  $\mathfrak{N} = \mathfrak{M}$ , whence  $N = M$ .

Now for (c)  $\implies$  (b): If  $\mathcal{F}$  is of finite type, then every point admits a special neighbourhood  $U$  over which we have a surjection:

$$\mathfrak{A}^p|_U \rightarrow \mathcal{F}|_U \rightarrow 0.$$

Since the kernel of this surjection is also of finite type, we in fact have an exact sequence (provided that the special open subset  $U$  is small enough):

$$\mathfrak{A}^q|U \rightarrow \mathfrak{A}^p|U \rightarrow \mathcal{F}|U \rightarrow 0.$$

The sheaf  $\mathcal{F}$  is thus, on every small enough special open subset, the cokernel of a morphism of quasi-coherent sheaves: so  $\mathcal{F}$  is locally quasi-coherent, and is thus quasi-coherent.

Finally, (a)  $\implies$  (c): If  $\mathcal{F}$  is coherent, then  $\mathcal{F}$  is of finite type. Furthermore, if  $\varphi: \mathcal{G} \rightarrow \mathcal{F}|U$  is a morphism, then we can always assume that  $U$  is a special open subset that is small enough such that we have a surjection  $\psi: \mathfrak{A}^p|U \rightarrow \mathcal{G} \rightarrow 0$ .

We then have the following diagram:

$$\begin{array}{ccccc} & & \mathfrak{A}^p|U & & \\ & & \downarrow \psi & \searrow \chi & \\ \text{Ker } \varphi & \longrightarrow & \mathcal{G} & \xrightarrow{\varphi} & \mathcal{F}|U \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

where  $\chi = \varphi \circ \psi$ .

The sheaves  $\mathfrak{A}^p|U$  and  $\mathcal{F}|U$  are coherent on  $U$ , and  $\chi$  is thus induced by a morphism  $\chi': \mathfrak{A}^p(U) \rightarrow \mathcal{F}(U)$ . This “resolution” of  $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$  extends to a resolution

$$\Gamma(U, \mathfrak{A})^q \rightarrow \Gamma(U, \mathfrak{A})^p \rightarrow \Gamma(U, \mathcal{F}).$$

This latter exact sequence allows us to complete the diagram and show that, on  $U$ ,  $\text{Ker } \varphi$  is the quotient of a sheaf of finite type:

$$\begin{array}{ccccc} \mathfrak{A}^q|U & \longrightarrow & \mathfrak{A}^p|U & \xrightarrow{\chi} & \mathcal{F}|U \\ \downarrow & & \downarrow \psi & & \updownarrow \\ \text{Ker } \varphi & \longrightarrow & \mathcal{G} & \xrightarrow{\varphi} & \mathcal{F}|U \\ \downarrow & & \downarrow & & \\ 0 & & 0 & & \end{array}$$

□

**Remark.** Claim (c) gives a characterisation of coherent sheaves, which does not require the base of special open subsets of  $V(A)$ . In fact, if  $V$  is an arbitrary topological space, and  $\mathfrak{A}$  a sheaf of rings on  $V$ , then the sheaves of  $\mathfrak{A}$ -modules that satisfy (c) are the objects of an abelian subcategory of the category of sheaves of  $\mathfrak{A}$ -modules (see Serre [4]). In the case considered here, the quasi-coherent sheaves are the inductive limits of coherent sheaves.

| p. 1-11



## 5 The maximal spectrum of a Jacobson ring

We are going to apply the above to algebraic geometry. For this, we denote by  $\Omega(A)$  the *maximal spectrum* of  $A$ , which is defined to be the topological subspace of  $V(A)$  consisting of the maximal ideals of  $A$ . We further suppose that  $A$  is a *Jacobson ring*, i.e. that every prime ideal is the intersection of maximal ideals (see [1] and [3]). The following proposition then holds true:

**Proposition.** *The following are equivalent:*

- (a)  $A$  is a Jacobson ring.
- (b) The correspondence  $U \mapsto U \cap \Omega(A)$  between open subsets of  $V(A)$  and of  $\Omega(A)$  is bijective.
- (c) The correspondence  $W \mapsto W \cap \Omega(A)$  between closed subsets of  $V(A)$  and of  $\Omega(A)$  is bijective.

*Proof.* The equivalence of (b) and (c) is trivial. On the other hand, the closed subsets of  $V$  correspond to the ideals of  $A$  that are intersections of prime ideals. The closed subsets of  $\Omega$  correspond to the ideals of  $A$  that are intersections of maximal ideals. The two families of ideals agree if and only if  $A$  is a Jacobson ring.  $\square$

Under this last hypothesis, the spaces  $V(A)$  and  $\Omega(A)$  thus have the same lattice of open subsets. It clearly follows that the sheaves on  $V$  and on  $\Omega$  “are in bijective correspondence”. In particular, we define a *special open subset* of  $\Omega$  to be the restriction of any special open subset of  $V$ , and a *quasi-coherent* (resp. *coherent*) *algebraic sheaf* on  $\Omega$  to be the restriction of any quasi-coherent (resp. coherent) algebraic sheaf on  $V$ . The properties of these sheaves on  $V$  extend (up to evident modifications) to their restrictions to  $\Omega$ .

In particular, every algebra of finite type over a field is a Jacobson ring. If  $k$  is an algebraically closed field,  $V$  an affine algebraic set over  $k$ , and  $A$  its coordinate ring, then  $V$  is homeomorphic to the maximal spectrum  $\Omega(A)$  of  $A$ . The sheaf of rings  $\mathfrak{A}|\Omega(A)$  is called the *sheaf of germs of regular functions* on  $V$ .

Similarly, a sheaf of modules over this sheaf of rings is said to be *algebraic* (resp. *quasi-coherent algebraic*, resp. *coherent algebraic*) if it is the restriction of a sheaf on  $V(A)$  with the named property.

| p. 1-12

## 6 Quasi-coherent sheaves on an algebraic variety

More generally, if  $X$  is an arbitrary algebraic set over  $k$ , then we define a *quasi-coherent* (resp. *coherent*) *algebraic sheaf* on  $X$  to be any sheaf  $\mathcal{F}$  such that there exists a cover of  $X$  by affine open subsets  $U_i$  such that  $\mathcal{F}|U_i$  is quasi-coherent (resp. coherent). By the [remark](#) that follows [Theorem 3](#), this is equivalent to saying that, for every affine open subset  $U$ , the sheaf  $\mathcal{F}|U$  is quasi-coherent (resp. coherent).

In particular, the sheaf of rings  $\mathfrak{A}$  such that, for every affine open subset  $U$ ,  $\mathfrak{A}(U)$  is the ring of regular functions on  $U$  (with the evident restrictions), is a coherent sheaf. We call it the *sheaf of germs of regular functions*. Every sheaf of modules over  $\mathfrak{A}$  is said to be *algebraic*.

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