Serre's theorem

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Translator's note.

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What follows is a translation of the French seminar talk:

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Part A
Affine algebraic sets and classical functors

1 Ringed topological spaces

From now on, and unless otherwise mentioned, the rings we consider will be assumed to be commutative, with a unit element, and *Noetherian*. We define a *ringed topological space* $(V, \mathscr{A})$ to be a topological space V endowed with the structure defined by the data of a sheaf of rings $\mathscr{A}$. If $(V, \mathscr{A})$ and $(W, \mathscr{B})$ are two ringed topological spaces, then we define a morphism from the first to the second by the data of: *p. 2-01*

[*] https://thosgood.com/translations

(a) a continuous map $\psi\colon V \to W$;

(b) for every open subset U of W, a ring homomorphism

$$\varphi_U\colon \mathscr{B}(U) \to \mathscr{A}(\varphi^{-1}(U))$$

that is compatible with the restriction maps.

The composition of two morphisms is defined in the evident way, and we speak of the category of ringed topological spaces. In what follows, V will almost always be a Zariski space. To such a V, we often associate a topological space $S(V)$, which is *the scheme of* V, and which is defined in the following way:

- the points of $S(V)$ are the closed irreducible subsets of V;
- the closed subsets of $S(V)$ are the sets $\mathscr{F}$, where F is a closed subset of V, and $\mathscr{F}$ denotes the set of closed irreducible subsets of V that are contained in F.

It is then clear that the correspondence $F \mapsto \mathscr{F}$ between closed subsets of V and closed subsets of $S(V)$ is bijective, that $S(V)$ is a Zariski space, and that every closed irreducible subset of $S(V)$ is the closure of a unique point. To every sheaf on V, we canonically associate a sheaf on $S(V)$. In particular, if $(V,\mathscr{A})$ is a ringed topological space, and if V is a Zariski space, then we denote by $(S(V),S(\mathscr{A}))$ the *scheme of* $(V,\mathscr{A})$, which is defined to be the ringed topological space given by $S(V)$ and the sheaf associated to $\mathscr{A}$. Of course, the scheme of $(S(V),S(\mathscr{A}))$ is isomorphic to $(S(V),S(\mathscr{A}))$. *p. 2-02*

If A is a Noetherian Jacobson ring, and $(\Omega(A),\mathscr{U})$ is the ringed topological space defined by its maximal spectrum, then the associated scheme is exactly $(V(A),\mathscr{U})$ (see Talk no. 1). If $(V,\mathscr{A})$ is an algebraic set endowed with the sheaf of germs of regular functions, then the associated scheme has been defined by Chevalley in [3] and [4]. In this case, V is the subspace of $S(V)$ given by the closed points, and $\mathscr{A}$ is the restriction of $S(\mathscr{A})$ to V. From now on, we will almost always restrict to the case of algebraic sets over an algebraically closed field.

2 Sheaves of ideals of an algebraic set

If V is an algebraic set, and $\mathscr{A}$ its sheaf of germs of regular functions, then the notion of quasi-coherent (resp. coherent) sheaves generalises the notion of a module (resp. module of finite type) over the ring of coordinates of an affine algebraic set. We similarly generalise the notions of support and dimension: if $\mathscr{F}$ is a coherent algebraic sheaf on V, then its *support* is the set of points x of V where the fibre $\mathscr{F}_x$ of $\mathscr{F}$ is not zero (if the fibre of $\mathscr{F}$ is zero at x, then it is zero on a neighbourhood of x, since $\mathscr{F}$ is coherent, and thus the support is closed). We then define the *dimension of* $\mathscr{F}$ to be the dimension of its support. When V is affine and the coherent sheaf $\mathscr{F}$ is associated to a module M of finite type, then the support of $\mathscr{F}$ is given by the set of maximal ideals of the coordinate ring that contain the annihilator of M.

The notion of an ideal generalises in the following way: we define a *sheaf of ideals* of $(V,\mathscr{A})$ to be any quasi-coherent (and thus coherent) subsheaf of $\mathscr{A}$. We then find the inevitable correspondence between ideals and closed subsets:

If $\mathscr{I}$ is a sheaf of ideals, then we associate to it the following closed subset $W(\mathscr{I})$ of V:

$x \in W(\mathscr{I})$ if and only if $\mathscr{I}_x$, the fibre of $\mathscr{I}$ at x, is a proper ideal of $\mathscr{O}_x$ (i.e. $\mathscr{I}_x \neq \mathscr{O}_x$), or if and only if the germs of the functions defined by $\mathscr{I}$ at x are zero. $W(\mathscr{I})$ is thus the support of $\mathscr{A}|\mathscr{I}$.

Conversely, if W is a closed subset of V, then we associate to it the following sheaf of ideals $\mathscr{I}(W)$:

The sections of $\mathscr{I}(W)$ on an open subset U are the regular functions defined on U that are zero on $U \cap W$. If $\mathscr{I}(U,x)$ denotes the inverse image in $\mathscr{A}(U)$ of the maximal ideal of $\mathscr{O}_x$, then $\Gamma(U,\mathscr{I}(W))$ is the intersection of the $\mathscr{I}(U,x)$ where x runs over $U \cap W$. If U is an affine open subset, then the sheaf of ideals $\mathscr{I}(W)|U$ is exactly the sheaf associated to the ideal $\Gamma(U,\mathscr{I}(W))$ of $\mathscr{A}(U)$, which shows that $\mathscr{I}(W)$ is coherent. *p. 2-03*

Proposition 1.

(a) The map $W \mapsto \mathscr{I}(W)$ gives a bijective correspondence between the closed subsets of V and the sheaves of ideals $\mathscr{I}$ that satisfy the following condition: for every $x \in V$, $\mathscr{I}_x$ is either equal to $\mathscr{O}_x$ or equal to an intersection of prime ideals of $\mathscr{O}_x$.

(b) The map $W \mapsto \mathscr{I}(W)$ associates the closed subsets whose connected components are all irreducible to the sheaves of prime ideals (for every x, $\mathscr{I}_x$ is either equal to $\mathscr{O}_x$ or equal to a prime ideal).

Proof. (a) It suffices to give a proof in the case where $(V,\mathscr{A})$ is an affine algebraic set. So let $A = \mathscr{A}(V)$, and $\mathfrak{a} = \Gamma(V,\mathscr{I}(W))$. We have already seen that $\mathfrak{a}$ is then the intersection of the prime ideals $\mathscr{I}(V,x)$, where x runs over W. So $\mathfrak{a}$ is an intersection of prime ideals. Since the correspondence between intersections of prime ideals of A and closed subsets of V is bijective, so too is the correspondence between ideals of A and sheaves of ideals of $(V,\mathscr{A})$; it remains only to show, conversely, that the sheaf associated to an intersection $\mathfrak{a}$ of prime ideals satisfies the condition of the proposition: this follows from the conservation properties of the prime decomposition under localisation.

(b) The proof is analogous.

□

3 The ring of rational functions of an algebraic set

We recall that, if V is an algebraic set, then we define a *rational function on V* to be a regular map f from an everywhere-dense open subset of V to the field of constants K. We further suppose that the domain of definition of f cannot be extended. The rational functions on V form an algebra $K(V)$ over K. If we denote by V_i the irreducible components of V, then $K(V)$ is isomorphic to the product $\prod_i K(V_i)$, with the isomorphism being the obvious one. Finally, if V is irreducible, and if U is an affine open subset of V, then $K(V)$ is exactly the field of fractions of the coordinate ring of U.

The sheaf $\mathscr{K}(V)$ of rational functions on V is then defined in the following way: to every open subset U of V, we associate the ring $K(U)$ of rational functions on U, with the restrictions being obvious. It is clear that this defines a quasi-coherent sheaf on V. Furthermore, if we denote by V_i the irreducible components of V, then $K(U)$ is exactly the product $\prod_{V_i \cap U \neq \varnothing} K(V_i)$. It thus follows that the sheaf $\mathscr{K}(V)$ is obtained in the following *p. 2-04* way: let $\mathscr{K}_i$ be the sheaf on V that is zero outside of V_i, and that is constant with fibre $K(V_i)$ on V_i. Then $\mathscr{K}(V)$ is the product of the sheaves $\mathscr{K}_i$.

4 Characterisation of affine algebraic sets

Let $(V,\mathscr{A})$ be an algebraic set endowed with its sheaf of rings, and let $A = \Gamma(V,\mathscr{A})$ be the coordinate ring of V. We know that we then have a canonical morphism $(V,\mathscr{A}) \to (S(V),S(\mathscr{A}))$, and that $(V,\mathscr{A})$ is affine if and only if $(S(V),S(\mathscr{A}))$ is the prime spectrum of an algebra over K, of finite type and with no nilpotent elements.

We will now define a morphism $(S(V),S(\mathscr{A})) \to (V(A),\mathscr{U})$. For this, let x be an arbitrary element of $S(V)$, and let $S(\mathscr{A})_x$ be the fibre of $S(\mathscr{A})$ at x: this an a local ring, and we have a restriction homomorphism $A \to S(\mathscr{A})_x$. We denote by $\mathfrak{p}_x$ the prime ideal of A given by the inverse image under this homomorphism of the maximal ideal of $S(\mathscr{A})_x$. It is an ideal of functions on V that are zero on the closed point x.

The map $\varphi\colon x \mapsto \mathfrak{p}_x$ is a continuous map from $S(V)$ to $V(A)$. Indeed, it suffices to show that the inverse image of a special open subset U_f of $V(A)$ is an open subset of $S(V)$. But $\varphi^{-1}(U_f)$ consists of points x of $S(V)$ such that the image f_x of f in $\mathscr{A}_x$ is invertible, and if f_x is invertible at x, then f_y is invertible for y in a neighbourhood of x. QED.

It follows from the above that $1/f$ is a section of $S(\mathscr{A})$ over $\varphi^{-1}(U_f)$, and so the canonical map from A to $\Gamma(\varphi^{-1}(U_f),S(\mathscr{A}))$ extends to a homomorphism:

$$\varphi_{U_f}\colon A_f \to \Gamma(\varphi^{-1}(U_f),S(\mathscr{A})).$$

The φ_{U_f} are compatible with the restriction maps, and, by "passing to the inductive limit", they define a morphism of ringed topological spaces: *p. 2-05*

$$\varphi\colon (S(V),S(\mathscr{A})) \to (V(A),\mathscr{U}).$$

It is clear that φ is an isomorphism if and only if $(V,\mathscr{A})$ is an affine algebraic set. The following theorem further examines this case:

Theorem (Serre's Theorem). *The following are equivalent:*

(a) $(V,\mathscr{A})$ *is an affine algebraic set.*

(b) If $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ *is an exact sequence of quasi-coherent sheaves, then the sections over V form an exact sequence.*

(c) If $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ *is an exact sequence of coherent sheaves, then the sections over V form an exact sequence.*

(d) There exist sections f_i *of* $\mathscr{A}$ *over V such that:*

- *the ideal generated by the* f_i *is equal to A; and*
- *the open subsets* V_{f_i} *of V, where the* f_i *are non-zero, are affine open subsets, and they cover V.*

Proof. It remains only to show that (c) implies (d), and that (d) implies (a).

For (c) $\implies$ (d), consider:

Lemma 1. *If $\mathscr{F}$ is a non-zero coherent sheaf, then $\Gamma(V,\mathscr{F})$ is non-zero.*

Proof. Indeed, the support of $\mathscr{F}$ contains a closed point, say x.

The fibre $\mathscr{F}_x$ of $\mathscr{F}$ is non-zero at x, and, if $\mathfrak{m}_x$ denotes the maximal ideal of $\mathscr{A}_x$, then $\mathscr{F}_x|\mathfrak{m}_x\mathscr{F}_x$ is non-zero (by the Nakayama lemma). The sheaf $\mathscr{G}$ that is zero away from x and "has the value" $\mathscr{F}_x|\mathfrak{m}_x\mathscr{F}_x$ at x is then coherent, and we clearly have a surjection:

$$\mathscr{F} \to \mathscr{G} \to 0.$$

It thus follows that we have an epimorphism from $\Gamma(V,\mathscr{F})$ to $\Gamma(V,\mathscr{G}) = \mathscr{F}_x|\mathfrak{m}_x\mathscr{F}_x$, and $\Gamma(V,\mathscr{F})$ is non-zero. □

With this proven, let f be a section of $\mathscr{A}$ over V (a regular function on V). If U is an arbitrary open subset of V, then we denote by U_f the set of points of U where f is non-zero. The existence of the cover (U_{f_i}) will be a consequence of:

Lemma 2. *If x is an arbitrary point of V, and U is an affine open subset of V containing x, then there exists a regular function f on V such that $V_f = U_f$, and such that V_f contains x.* p. 2-06

Proof. Let W be the complement of U in V, and let $\mathscr{I}(W)$ (resp. $\mathscr{I}(W,x)$) be the sheaf of germs of functions that are zero on W (resp. on W and x). We then have an exact sequence:

$$0 \to \mathscr{I}(W,x) \to \mathscr{I}(W) \to \mathscr{I}(W)|\mathscr{I}(W,x) \to 0$$

and the sheaf $\mathscr{I}(W)|\mathscr{I}(W,x)$ is clearly non-zero. So there exists a section f of $\mathscr{I}(W)$ that does not belong to $\Gamma(\mathscr{I}(W,x))$; this is the desired function. □

The quasi-compactness of V, along with Lemma 2, imply the existence of an affine cover of V by a finite number of V_{f_i}. It remains only to show that the ideal generated by the f_i is A. But if U is an affine open subset, then the U_{f_i} cover U, and it follows that restrictions of the f_i to $\mathscr{A}(U)$ generate the unit ideal. In other words, if there are p sections f_i, and if $\mathscr{A}^p$ denotes the direct sum of p sheaves, each isomorphic to $\mathscr{A}$, then the morphism from $\mathscr{A}^p$ to $\mathscr{A}$ defined by the f_i is surjective. The same is true for the morphism $A^p \to A$ defined by the f_i.

For (d) $\implies$ (a): we will show that the morphism

$$\varphi\colon (S(V),S(\mathscr{A})) \to (V(A),\mathscr{U})$$

is an isomorphism, and that A is an algebra of finite type with no nilpotent elements.

Note first of all that, if $f \in A$ is a regular function on V, then the coordinate ring of V_f is exactly A_f. Indeed, if A_i denotes the coordinate ring of V_{f_i}, and $A_{ij} = A_{if_j} = A_{jf_i}$ the coordinate ring of $V_{f_i} \cap V_{f_j} = V_{f_i \cdot f_j}$, then we have the famous exact sequence (see Talk no. 1):

$$0 \to A \to \prod_i A_i \to \prod_{i \neq j} A_{ij}$$

whence (by the exactness of the functor $M \mapsto M_f$) we have the exact sequence:

$$0 \to A_f \to \prod_i (A_i)_f \to \prod (A_{ij})_f.$$

But the $(A_i)_f$ and $(A_{ij})_f$ are clearly the coordinate rings of $V_{f_i} \cap V_f$ and $V_{f_i} \cap V_{f_j} \cap V_f$; thus A_f is coordinate ring of V_f. *p. 2-07*

From this we see that the map $S(V) \to V(A)$ is injective, and that it induces an isomorphism

$$(S(V_{f_i}), \mathscr{A} | S(V_{f_i})) \to (V(A)_{f_i}, \mathscr{U} | V(A)_{f_i}).$$

Also, the $V(A)_{f_i}$ cover $V(A)$, since the f_i generate the unite ideal. By gluing the pieces together, we thus obtain an isomorphism

$$(S(V), S(\mathscr{A})) \to (V(A), \mathscr{U})$$

and it remains only to show that A is of finite type and has no nilpotent elements.

But A is an algebra of functions, and has no nilpotent elements. Now, for all i, let a_{i_k} be a finite number of elements of A such that the images of f_i and a_{i_k} in A_{f_i} generate the ring A_{f_i}. Finally, let $a_i \in A$ be such that $\sum a_i f_i = 1$ in A. We will show that the f_i, a_{i_k}, and a_i all together generate the ring A:

Indeed, by taking a suitable power of the formula $\sum a_i f_i = 1$, we obtain formulas $\sum a_i(m) f_i^m = 1$, where the $a_i(m)$ can be expressed in terms of the a_i and f_i. If now x is an element of A, then its image in A_{f_i} can be written as $x_i | f_i^p$ for some p large enough. It then follows (see Talk no. 1) that, for n large enough, we have

$$x = \sum_i a_i(n+p) x_i f_i^n$$

which finishes the proof. □

Corollary. *If V is an affine algebraic set, U an open subset of V, A the coordinate ring of V, and $\mathscr{U}(U)$ the coordinate ring of U, then the following are equivalent:*

(a) U is an affine open subset.

(b) If M is an A-module, then the canonical homomorphism

$$M \otimes_A \mathscr{U}(U) \to \mathscr{M}(U)$$

is bijective.

Furthermore, if one of these equivalent properties is satisfied, then $\mathscr{U}(U)$ is A-flat. *p. 2-08*

Proof. (a) $\Longrightarrow$ (b): Indeed, $M \mapsto \mathscr{M}(U)$ is then a right-exact functor in M, where M runs over all A-modules. In particular, if

$$L_1 \to L_0 \to M \to 0$$

is a resolution of M by free A-modules, then we have the diagram

$$\begin{array}{ccccccc} L_1 \otimes_A \mathscr{U}(U) & \longrightarrow & L_0 \otimes_A \mathscr{U}(U) & \longrightarrow & M \otimes_A \mathscr{U}(U) & \longrightarrow & 0 \\ \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} & & \\ \mathscr{L}_1(U) & \longrightarrow & \mathscr{L}_0(U) & \longrightarrow & \mathscr{M}(U) & \longrightarrow & 0 \end{array}$$

Since v and w are isomorphisms, so too is u.

(b) $\implies$ (a): We will show that, if

$$0 \to \mathscr{F}' \to \mathscr{G}' \to \mathscr{H}' \to 0$$

is an exact sequence of coherent sheaves on U, then the sections on U give an exact sequence.

But there exists an exact sequence

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$$

of coherent sheaves on V such that $\mathscr{F}|U = \mathscr{F}'$, $\mathscr{G}|U = \mathscr{G}'$, and $\mathscr{H}|U = \mathscr{H}'$ (see [1], with the proof in the appendix of [2]). So, if M, N, and P are the modules associated to $\mathscr{F}$, $\mathscr{G}$, and $\mathscr{H}$, then the sequence

$$M \otimes_A \mathscr{U}(U) \to N \otimes_A \mathscr{U}(U) \to P \otimes_A \mathscr{U}(U) \to 0$$

is exact, which proves that the homomorphism

$$\mathscr{G}'(U) = N \otimes_A \mathscr{U}(U) \to \mathscr{H}'(U) = P \otimes_A \mathscr{U}(U)$$

is surjective. □

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