

Divisors in algebraic geometry

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Translator's note.

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In the first part of this talk, we will prove a theorem of Serre on complete varieties [6], following the methods of Grothendieck [4]. The second part is dedicated to generalities on divisors. In the literature, we often call the divisors studied here "locally principal" divisors.

The algebraic spaces considered here are defined over an algebraically closed field K . By "variety", we mean an irreducible algebraic space. If X is an algebraic space, we denote by $\mathcal{O}(X)$, $\mathcal{R}(X)$, etc. (or simply \mathcal{O} , \mathcal{R} , etc.) the structure sheaf, of regular functions, etc. on X (to define $\mathcal{R}(X)$ we assume that X is a variety). By "coherent sheaf" on X , we mean a coherent sheaf of \mathcal{O} -modules on X .

1 Preliminaries

[4, 5, 6]

If M is a module over an integral ring A (commutative and with 1), then we say that an element $m \in M$ is a *torsion element* if there exists some non-zero $a \in A$ such that $a \cdot m = 0$. We say that M is a *torsion module* (resp. *torsion-free module*) if every element of M is a torsion element (resp. if $M \neq 0$ and no non-zero element of M is a torsion element). The torsion elements of M form a torsion submodule of M (denoted by $T(M)$); if $M \neq 0$, then

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$M/T(M)$ is a torsion-free module. If M is a torsion module of finite type over A , then the ideal $\text{ann} M$ of A (the ideal of A given by the elements $a \in A$ such that $aM = 0$) is non-zero.

Let X be an algebraic space and \mathcal{F} a sheaf of \mathcal{O} -modules on X . We define $\text{supp } \mathcal{F}$ to be the set of points $x \in X$ such that $\mathcal{F}_x \neq 0$. If \mathcal{F} is coherent, then $\text{supp } \mathcal{F}$ is a closed subset of X . If X is affine, then $\text{supp } \mathcal{F}$ is the set defined by the ideal $\text{ann} H^0(X, \mathcal{F})$ of the affine algebra $H^0(X, \mathcal{O})$, where $H^0(X, \mathcal{F})$ is considered as a module over $H^0(X, \mathcal{O})$.

A sheaf \mathcal{F} of \mathcal{O} -modules on a variety X is said to be a *torsion sheaf* (resp. *torsion-free sheaf*) if, for every $x \in X$, the module \mathcal{F}_x over the ring \mathcal{O}_x is a torsion module (resp. torsion-free module).

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Proposition 1. *If \mathcal{F} is a coherent sheaf on a variety X , then there exists a coherent subsheaf $T(\mathcal{F})$ of \mathcal{F} (and only one) such that $(T(\mathcal{F}))_x = T(\mathcal{F}_x)$.*

Proof. The uniqueness is trivial. The existence is a consequence of the fact that, if X is affine, then $T(\mathcal{F}_x)$ is given by localisation of the module $T(H^0(X, \mathcal{F}))$ with respect to the maximal ideal of $H^0(X, \mathcal{O})$ that defines x . \square

Corollary. *If $\mathcal{F} \neq 0$ then $\mathcal{F}/T(\mathcal{F})$ is a torsion-free coherent sheaf.**

Proposition 2. *If \mathcal{F} is a coherent sheaf on the variety X , then $\text{supp } \mathcal{F} \neq X$ if and only if \mathcal{F} is a torsion sheaf.*

Proof. This is a trivial consequence of the fact that, if U is an affine open subset, then $\text{supp } \mathcal{F} \cap U$ is defined by the ideal $\text{ann} H^0(U, \mathcal{F})$ of $H^0(U, \mathcal{O})$, where $H^0(U, \mathcal{F})$ is considered as a module over $H^0(U, \mathcal{O})$. \square

Proposition 3. *If \mathcal{F} is a torsion-free coherent sheaf on a variety X , with $\mathcal{F} \subset \mathcal{R}^n$, then there exists a coherent sheaf $\mathcal{I} \neq 0$ of ideals of \mathcal{O} such that $\mathcal{I} \cdot \mathcal{F} \subset \mathcal{O}^n$.*

Proof. Let \mathcal{I}_x be the ideal $[\mathcal{O}_x^n : \mathcal{F}_x]$ of \mathcal{O}_x , i.e. the ideal of elements i_x of \mathcal{O}_x such that $i_x \mathcal{F}_x \subset \mathcal{O}_x^n$. Since \mathcal{F}_x is of finite type over \mathcal{O}_x , we know that $\mathcal{I}_x \neq 0$. If we take an affine open subset U of X , then we can prove that \mathcal{I}_x is given by localisation of the ideal $[H^0(U, \mathcal{O}^n) : H^0(U, \mathcal{F})]$ of $H^0(U, \mathcal{O})$ by the maximal ideal of $H^0(U, \mathcal{O})$ that defines x . Thus $\{\mathcal{I}_x\}_{x \in X}$ defines a coherent sheaf \mathcal{I} of ideals of \mathcal{O} such that $\mathcal{I} \cdot \mathcal{F} \subset \mathcal{O}^n$. \square

Let \mathcal{F} be a torsion-free coherent sheaf on a variety X . Then the canonical homomorphism $\mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}} \mathcal{R}$ is injective. The sheaves \mathcal{R} and $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{R}$ are locally constant sheaves, and thus constant ([5, page 229]). We can then identify $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{R}$ with a vector space of finite dimension over \mathcal{R} (we identify the field of rational functions with the sheaf \mathcal{R} since \mathcal{R} is constant). We call this dimension the *rank* of \mathcal{F} , and we can then consider \mathcal{F} as a subsheaf of \mathcal{R}^n , where $n = \text{rank } \mathcal{F}$.

Proposition 4. *Under the same hypotheses as in Proposition 3, there exists a coherent sheaf $\mathcal{I} \neq 0$ of ideals of \mathcal{O} such that $\mathcal{I} \cdot \mathcal{F} \subset \mathcal{O}^n$, where $n = \text{rank } \mathcal{F}$; then $\mathcal{O}^n/(\mathcal{I} \cdot \mathcal{F})$ and $\mathcal{F}/(\mathcal{I} \cdot \mathcal{F})$ are torsion sheaves.*

*[Trans.] The condition that $\mathcal{F} \neq 0$ is unnecessary, but we include it here since it is in the original. Note that the zero sheaf is indeed a torsion-free sheaf, otherwise any coherent torsion sheaf \mathcal{F} provides a counterexample to this corollary.

Proof. The proof is immediate. \square

If Y is a closed subset of an algebraic space X , then we denote by \mathcal{I}_Y the coherent sheaf of ideals of \mathcal{O} defined by Y .

Proposition 5. *Let Y be a closed subset of an algebraic space X , and \mathcal{F} a coherent sheaf on X , with $\text{supp } \mathcal{F} \subset Y$; then there exists an integer k such that $\mathcal{I}_Y^k \mathcal{F} = 0$.*

Proof. We can reduce to the case where X is affine, since there exists a finite cover of X by affine opens. In this case, the hypothesis implies that the set defined by the ideal $\text{ann}H^0(X, \mathcal{F})$ is contained in Y . This implies, as is well known, that $\text{ann}H^0(X, \mathcal{F}) \supset \mathcal{I}_Y^k$. \square

Proposition 6. *Let \mathcal{F} be a coherent sheaf of fractional ideals on a variety X (i.e. a coherent subsheaf of \mathcal{R}) such that, for every x outside of a closed subset Y of X , \mathcal{F}_x is an ideal of \mathcal{O}_x . Then there exists an integer k such that $\mathcal{I}_Y^k \cdot \mathcal{F} \subset \mathcal{O}$.*

Proof. By Proposition 3 and the hypothesis, there exists a coherent sheaf \mathcal{J} of ideals of \mathcal{O} such that $\mathcal{J}_x = \mathcal{O}_x$ if $x \notin Y$, and such that $\mathcal{J} \cdot \mathcal{F} \subset \mathcal{O}$. Thus $\text{supp}(\mathcal{O}/\mathcal{J}) \subset Y$, and, by Proposition 5, there exists an integer k such that $\mathcal{I}_Y^k(\mathcal{O}/\mathcal{J}) = 0$. This implies that $\mathcal{I}_Y^k \subset \mathcal{J}$. \square

2 Dévissage theorem

Let \mathcal{C} be an abelian category, and \mathcal{C}' a subcategory of objects of \mathcal{C} . We say that \mathcal{C}' is *left exact in \mathcal{C}* if

1. every subobject of an object of \mathcal{C}' is in \mathcal{C}' ;
2. for every exact sequence $0 \rightarrow \mathcal{A}' \rightarrow \mathcal{A} \rightarrow \mathcal{A}'' \rightarrow 0$ in \mathcal{C} , the object \mathcal{A} is in \mathcal{C}' if the other two objects are in \mathcal{C}' .²

Let X be an algebraic space. We denote by $\mathcal{C}(X)$ the abelian category of coherent sheaves on X . If Y is a closed subset of X , then a coherent sheaf on Y has a canonical extension to a coherent sheaf on X (extending by 0 outside of Y), and so we can consider $\mathcal{C}(Y)$ as a subcategory of $\mathcal{C}(X)$. With this notation, we have the following theorem:

Theorem (Dévissage). *Let \mathcal{D} be a left-exact subcategory of $\mathcal{C}(X)$ that has the following property: for every closed irreducible subset Y of X , there exists a coherent sheaf \mathcal{M}_Y of $\mathcal{C}(Y)$ that belongs to \mathcal{D} , and that is torsion-free as a sheaf on Y . Then $\mathcal{D} = \mathcal{C}(X)$.*

Proof. The proof works by induction on the dimension of X . If $\dim X = 0$, then X consists of a finite number of points P_1, \dots, P_r , and a coherent sheaf on X can be identified with a system $\{N_i\}_{i=1, \dots, r}$, where N_i is a vector space of finite dimension over K . Thus the sheaf \mathcal{M}_{P_i} on P_i that we have, by hypothesis, is a vector space of finite dimension over K . By the axioms of a left-exact subcategory, it is trivial to show that every system $\{N_i\}_{i=1, \dots, r}$,

²The axioms here that define a left-exact subcategory are slightly stronger than those of Grothendieck [4].

where N_i is a vector space of finite dimension over K , considered as a coherent sheaf on X , belongs to \mathcal{D} .

Now assume that we have proven the theorem for all dimensions $\leq (n-1)$. Let $\dim X = n$. Let Y be a closed subset of X such that $\dim Y \leq (n-1)$. We can easily show that $\mathcal{D} \cap \mathcal{C}(Y)$ is a left-exact subcategory of $\mathcal{C}(Y)$ that satisfies the hypotheses of the theorem. So, by the induction hypothesis, $\mathcal{D} \supset \mathcal{C}(Y)$.

We will now prove that, if \mathcal{F} is a coherent sheaf on X with $\text{supp } \mathcal{F} = Y$, then $\mathcal{F} \in \mathcal{D}$. If $\mathcal{I}_Y \cdot \mathcal{F} = 0$, then $\mathcal{F} \in \mathcal{C}(Y)$, and, by the above, $\mathcal{F} \in \mathcal{D}$. No matter what, by [Proposition 5](#), there exists an integer $k \geq 1$ such that $\mathcal{I}_Y^k \mathcal{F} = 0$. We will complete the proof by induction on k . Suppose that that claim has been proven for every coherent sheaf \mathcal{G} on X such that $\mathcal{I}_Y^{k-1} \mathcal{G} = 0$. For \mathcal{F} , we have an exact sequence

$$0 \rightarrow \mathcal{I}_Y \cdot \mathcal{F} \rightarrow \mathcal{F} \rightarrow \mathcal{F}/(\mathcal{I}_Y \cdot \mathcal{F}) \rightarrow 0.$$

The sheaf $\mathcal{I}_Y \mathcal{F}$ is annihilated by \mathcal{I}_Y^{k-1} , and the sheaf $\mathcal{F}/(\mathcal{I}_Y \mathcal{F})$ is annihilated by \mathcal{I}_Y . Thus $\mathcal{I}_Y \mathcal{F}$ and $\mathcal{F}/(\mathcal{I}_Y \mathcal{F})$ belong to \mathcal{D} . This implies that $\mathcal{F} \in \mathcal{D}$.

Suppose that X is a variety, and that \mathcal{F} is a torsion-free sheaf on X . We can consider \mathcal{F} as a coherent subsheaf of \mathcal{R}^n , where $n = \text{rank } \mathcal{F}$, and, by [Proposition 4](#), there then exists a coherent sheaf of ideals \mathcal{I} such that $\mathcal{I} \cdot \mathcal{F} \subset \mathcal{O}^n$, and such that the sheaves $\mathcal{F}/(\mathcal{I} \mathcal{F})$ and $\mathcal{O}^n/(\mathcal{I} \mathcal{F})$ are torsion sheaves. Since $\mathcal{F}/(\mathcal{I} \mathcal{F})$ is a torsion sheaf, $\mathcal{F}/(\mathcal{I} \mathcal{F}) \in \mathcal{D}$; thus $\mathcal{F} \in \mathcal{D}$ if and only if $\mathcal{I} \mathcal{F} \in \mathcal{D}$. Analogously, $\mathcal{I} \mathcal{F} \in \mathcal{D}$ if and only if $\mathcal{O}^n \in \mathcal{D}$, and, by the axioms of an exact subcategory, if and only if $\mathcal{O} \in \mathcal{D}$. Thus $\mathcal{F} \in \mathcal{D}$ if and only if $\mathcal{O} \in \mathcal{D}$. If we repeat the same argument for the torsion-free sheaf \mathcal{M}_X , which we have by hypothesis, then we see that $\mathcal{O} \in \mathcal{D}$, which implies that $\mathcal{F} \in \mathcal{D}$. | p. 4-05

Suppose again that X is a variety, but now that \mathcal{F} is an arbitrary coherent sheaf. We will show that $\mathcal{F} \in \mathcal{D}$. We can assume that $\mathcal{F} \neq 0$, and we then have

$$0 \rightarrow T(\mathcal{F}) \rightarrow \mathcal{F} \rightarrow \mathcal{F}/T(\mathcal{F}) \rightarrow 0$$

where $T(\mathcal{F})$ is a torsion sheaf, and $\mathcal{F}/T(\mathcal{F})$ is a torsion-free sheaf. By [Proposition 2](#), $\text{supp } T(\mathcal{F}) \neq X$, and, since X is a variety, $\dim \text{supp } T(\mathcal{F}) < \dim X$. We then have, by the induction hypothesis, that $T(\mathcal{F}) \in \mathcal{D}$, and we have just proven that $\mathcal{F}/T(\mathcal{F}) \in \mathcal{D}$. Thus $\mathcal{F} \in \mathcal{D}$.

Now let X be an arbitrary algebraic space, and X_1, \dots, X_p its irreducible components. If \mathcal{F} is a coherent sheaf on X , then $\mathcal{F}/(\mathcal{I}_{X_i} \mathcal{F})$ can be identified with a sheaf on the variety X_i (where \mathcal{I}_{X_i} is the sheaf of ideals of $\mathcal{O}(X)$ determined by X_i), and, by the above, $\mathcal{F}/(\mathcal{I}_{X_i} \mathcal{F}) \in \mathcal{D}$. Thus the sheaf $\mathcal{G} = \sum_{i=1}^p \mathcal{F}/(\mathcal{I}_{X_i} \mathcal{F})$ belongs to \mathcal{D} . We have a canonical homomorphism $\varphi: \mathcal{F} \rightarrow \mathcal{G}$. The image of φ is a coherent subsheaf of \mathcal{G} , and so the image of φ belongs to \mathcal{D} .

We know that $\text{supp } \text{Ker } \varphi \subset \bigcup_{i \neq j} X_i \cap X_j$, and so $\dim \text{supp } \text{Ker } \varphi < \dim X$, and, by the induction hypothesis, $\text{Ker } \varphi \in \mathcal{D}$. Thus $\mathcal{F} \in \mathcal{D}$, and the theorem is proven. \square

Corollary (Serre's Theorem). *If \mathcal{F} is a coherent sheaf on a complete algebraic space X , then $H^0(X, \mathcal{F})$ is a vector space of finite dimension over K .*

Proof. We take \mathcal{D} to be the category of all coherent sheaves \mathcal{F} on X such that $H^0(X, \mathcal{F})$ is of finite dimension over K . We can prove that \mathcal{D} is a left-exact subcategory of $\mathcal{C}(X)$. Also,

we know that, if Y is an irreducible closed subset of X , then Y is a complete variety. Thus the coherent sheaf $\mathcal{O}(Y)$ on Y is a torsion-free sheaf with the property that $H^0(Y, \mathcal{O}(Y)) \cong K$, and so $H^0(X, \mathcal{O}(Y)) = H^0(Y, \mathcal{O}(Y))$ is of finite dimension over K (we denote also by $\mathcal{O}(Y)$ the canonical extension of $\mathcal{O}(Y)$ to X). By [the theorem](#), the corollary is proven. \square

3 Divisors (Generalities)

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Let X be an algebraic variety, and $\mathcal{R}^\times(X)$ and $\mathcal{O}^\times(X)$ (or simply \mathcal{R}^\times and \mathcal{O}^\times) the constant sheaf on X of non-zero rational functions and the sheaf on X of invertible regular functions (respectively). The sheaves \mathcal{R}^\times and \mathcal{O}^\times , endowed with their multiplicative structure, are sheaves of abelian groups.

A *divisor* D on X is a section of the quotient sheaf $\mathcal{R}^\times/\mathcal{O}^\times$. An element of \mathcal{R}^\times that is a representative of the value $D(x)$ of D at x is called a *definition function of D at x* . More generally, a function $f \in \mathcal{R}^\times$ is called a *definition function of D in an open subset U* if, for all $x \in U$, f is a representative of $D(x)$; then f is determined up to an invertible regular function on U . Since we can locally lift a section of $\mathcal{R}^\times/\mathcal{O}^\times$ to a section of \mathcal{R}^\times , a divisor D is determined by the following data: a cover $\{U_i\}$ by open subsets, and non-zero rational functions f_i on U_i such that, on $U_i \cap U_j$, $f_{ij} = f_i/f_j$ is an invertible regular function. We have that $f_{ij}f_{jk}f_{ki} = 1$ on $U_i \cap U_j \cap U_k$, and, as is well known, this allows us to construct a locally trivial fibre bundle with K^\times as the structure group; it is easy to see that this fibre bundle is determined up to equivalence [7]. We also know that the coherent sheaves of fractional ideals (i.e. the coherent subsheaves of \mathcal{R}) that are generated by f_i and f_j (respectively) agree on $U_i \cap U_j$, and do not depend on the choice of definition functions of D in U_i and U_j . This implies that the divisor D canonically determines a coherent sheaf of locally principal fractional ideals. We can easily see that the converse is true, and this gives us an equivalent definition of a divisor [1].

A divisor D is said to be *positive* if, for each $x \in X$, $D(x) \in \mathcal{O}_x/\mathcal{O}_x^\times$ (i.e. if all the definition functions of D at x are regular functions in x).

Since $\mathcal{R}^\times/\mathcal{O}^\times$ is a sheaf of abelian groups on X , there is a canonical structure of an abelian group on the set of divisors on X ; this group is called the *group of divisors on X* . The composition law in this group is written additively, and the identity element in this group is thus called the *zero divisor*, and is denoted by (0) .

If f is a non-zero rational function on X , then it defines a divisor $\text{div} f$ by the data $(\text{div} f)(x) = \text{Im} f \subseteq \mathcal{R}^\times/\mathcal{O}_x^\times$. The divisors obtained in this way are called *principal divisors*, and form a subgroup of the group of divisors on X ; the quotient group is called the *group of classes of divisors on X* . Two divisors D_1 and D_2 are said to be equivalent if they are equivalent module the group of principal divisors; we write $D_1 \sim D_2$. We have seen that a divisor defines, up to equivalence, a locally trivial algebraic fibre bundle with structure group K^\times . On the other hand, it is easy to see that a locally trivial algebraic fibre bundle with K^\times as its structure group defines, up to equivalence, a divisor [7]. Thus the group of classes of divisors on X is equal to $H^1(X, \mathcal{O}^\times)$, the group of classes of equivalent algebraic fibre bundles with K^\times as their structure group.

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We can define, in an analogous way, an *additive divisor* on a variety X as a section of the sheaf \mathcal{R}/\mathcal{O} (the divisors defined above are called *multiplicative divisors*, or simply *divisors*). The additive divisors form an abelian group, and even a vector space over K . An additive divisor is determined by the following data: a cover $\{U_i\}$ of X by open subsets,

and rational functions f_i on U_i such that $f_{ij} = f_i - f_j$ is a regular function on $U_i \cap U_j$. We can define, as for (multiplicative) divisors, the notions of definition functions of an additive divisor, equivalence between two additive divisors, etc. We find, for example, that $H^1(X, \mathcal{O})$ is equal to the group of classes of additive divisors on X .

Let D be a (multiplicative) divisor on X . We define $\text{supp} D$ to be the set of points $x \in X$ such that $D(x)$ is not the identity element in $\mathcal{R}^\times / \mathcal{O}_x^\times$, i.e. such that every definition function of D at x is either not defined at x or takes the value 0 at x .

Proposition 7. *The support of a divisor D on a variety X is a closed subset $\neq X$ of X , and $D = 0$ if and only if the support is empty.*

Proof. The latter claim is trivial. For the former, we prove that the set E of points $x \in X$ such that every definition function of D at x belongs to \mathcal{O}_x^\times is a non-empty open subset; indeed, if we take a definition function g of D at x , then it is also a definition function of D in an open subset U that contains x . By hypothesis, if $x \in E$, then g is regular at x and $g(x) \neq 0$, and we can choose U such that g is regular and invertible on U , which proves that E is open. \square

| p. 4-08

Proposition 8. *If D is a divisor on a normal variety X , then $\text{supp} D$ is a union of hypersurfaces (i.e. of closed subvarieties of codimension 1).*

Proof. If f is a function on a normal variety Y , we know that, if f is not defined at $x \in Y$, then x belongs to a variety of poles or zeros of f (i.e. to an irreducible component of the closure of the set of points $x \in Y$ such that $f(x) \in \{0, \infty\}$). So, if we take f to be a definition function for D in some open subset $U \subset X$, then $\text{supp} D \cap U$ is the union of the pole and zero varieties of f in U , and we know that these varieties are of codimension 1 ([2, chapitre III]). \square

Remark. If X is not normal, then the support of a divisor D on X is not necessarily of codimension 1. It is easy to define an affine variety X of dimension > 1 that is normal everywhere except at a single point x_0 (for example, the point (a, ab, b^2, b^3) in the four-dimensional space K^4). There exists a function u that is everywhere defined on X , and which is entire on the local ring of x_0 , but which is not contained in this ring; by adding, if necessary, a constant, we can assume that x_0 is not a zero of u . There is then an open neighbourhood X' of x_0 such that the divisor of the function induced by u on X' has support equal to the single point x_0 .

Suppose that X is a normal variety, and D is a divisor on X . Let S be a hypersurface of X . If f is a definition function of D at $x \in S$, then the order of f on S ([2]) does not depend on the choice of f , nor on $x \in S$. We denote this integer by $\text{ord}_S D$. It is easy to see that $\text{ord}_S D = 0$ if and only if $S \not\subset \text{supp} D$. If we now take the formal combination $C = \sum_S (\text{ord}_S D) \cdot S$, where S runs over the set of all hypersurfaces of X , then C is a cycle of codimension 1, and we call it the *associated cycle* of the divisor D .

Proposition 9. *Let X be a normal variety. The map that sends a divisor D to its associated cycle of codimension 1 is an injective homomorphism from the group of divisors on X to the group of cycles of codimension 1.*

Proof. The proof is trivial. □

Proposition 10. *If X is further a non-singular variety, then the homomorphism that sends a divisor to its associated cycle is bijective.*

| p. 4-09

Proof. It suffices to show that, for every hypersurface S , there exists a divisor D such that the cycle $1 \cdot S$ is the cycle associated to D . Since X is non-singular, for every $x \in X$, the local ring \mathcal{O}_x is factorial [3]; thus, for every $x \in S$, S is defined by one single equation in a neighbourhood of x . So there exists a cover $\{U_i\}_{i=1,\dots,p}$ of S by open subsets U_i of X , and, for each i , a regular function f_i on U_i that is non-zero outside of $U_i \cap S$ in U_i with $\text{ord}_S f_i = 1$. It then follows that f_i/f_j is an invertible regular function in $U_i \cap U_j$. Now take the cover $\{U_i\}_{i=0,\dots,p}$, where $U_0 = CS$, and take $f_0 = 1$. It is easy to see that the divisor D for which f_i is a definition function of D on U_i is such that the cycle associated to D is $1 \cdot S$. So the proposition is proven. □

Remark. The proposition is not necessarily true if X is non-singular. For example, for the cone $xy - zw = 0$ in K^4 , the cycle defined by $x = z = 0$ is not a cycle associated to any divisor.

References

- [1] Cartier, P. Diviseurs et dérivations en géométrie algébrique. (*Thèse Sc. math. Paris.* 1958) (To appear in *Bull. Soc. math. France*).
- [2] Chevalley, C. Fondements de la Géométrie algébrique Paris, Secrétariat mathématique, 1958, multigraphed. (Class taught at the Sorbonne in 1957–58).
- [3] Godement, R. Propriétés analytiques des localités In *Séminaire Cartan-Chevalley*, vol. **8**, 1955–56. (Talk number 19).
- [4] Grothendieck, A. Sur les faisceaux algébriques et les faisceaux analytiques cohérents. In *Séminaire H. Cartan*, vol. **9**. (Talk number 2).
- [5] Serre, J.-P. Faisceaux algébriques cohérents. *Ann. Math.* **61** (1955), 197–279.
- [6] Serre, J.-P. Sur la cohomologie des variétés algébriques. *J. Math. pures et appl.* **36** (1957), 1–16.
- [7] Weil, A. Fibre spaces in algebraic geometry (Notes taken by A. Wallace, 1952) Chicago, University of Chicago, 1955.