

Families of complex spaces and the foundations of analytic geometry

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Translator's note

This page is a translation into English of the following:

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<i>[Translator] According to the complete list of talks, the notes from the first talk of the 1960/61 Séminaire Henri Cartan — “Fibrés en tores complexes” (also given by Adrien Douady) — were not copied, and thus seem to be lost to the past. What follows is a translation of the next three talks in this seminar series.</i>	

2. Mixed manifolds and mixed spaces

I. Category of models

Let B be a topological space. We define the category \mathcal{S}_B^n in the following manner: the objects of \mathcal{S}_B^n are the open subsets of $B \times \mathbb{C}^n$, and a morphism $f: U \rightarrow U'$ from an open subset $U \subset B \times \mathbb{C}^n$ to an open subset $U' \subset B \times \mathbb{C}^n$ is a continuous map $f: U \rightarrow U'$ satisfying the following two conditions:

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1. the diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & U' \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ B & \xlongequal{\quad} & B \end{array}$$

commutes, where π_1 denotes the projection of $B \times \mathbb{C}^n$ to B ; and

2. for all $x \in B$, the map $f_x: U_x \rightarrow U'_x$ is holomorphic, where

$$U_x = \{z \in \mathbb{C}^n \mid (x, z) \in U\}$$

(and similarly for U').

If B is endowed with the structure of a \mathcal{C}^∞ manifold (resp. an \mathbb{R} -analytic manifold, resp. \mathbb{C} -analytic manifold), then we obtain a category $\mathcal{C}^\infty \mathcal{S}_B$ (resp. $\mathbb{R} \mathcal{S}_B$, resp. $\mathbb{C} \mathcal{S}_B$) by requiring the morphisms to be \mathcal{C}^∞ (resp. \mathbb{R} -analytic, resp. \mathbb{C} -analytic).

More generally, if $f_1: B \rightarrow B'$ is a continuous map from one topological space to another, then a *morphism of \mathcal{S}_{f_1}* is a continuous map f from an object U of \mathcal{S}_B to an object U' of $\mathcal{S}_{B'}$ such that

1. the diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & U' \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ B & \xrightarrow{f_1} & B' \end{array}$$

commutes; and

2. $f_x: U_x \rightarrow U'_{f_1(x)}$ is holomorphic for all $x \in B$.

If f_1 is a \mathcal{C}^∞ map from one \mathcal{C}^∞ manifold to another, then f will be a morphism of $\mathcal{C}^\infty \mathcal{S}_{f_1}$ if, further, it is a \mathcal{C}^∞ map (resp. ...). We thus obtain, for every category of topological spaces, a fibred category \mathcal{S}^n (resp. $\mathcal{C}^\infty \mathcal{S}^n$, resp. ...).

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II. The definition of mixed spaces and mixed varieties

1. First definition

Let B and V be separated spaces, and let $\pi: V \rightarrow B$ be a continuous map. The structure of a *mixed space* over B is defined on V by a system of charts $\varphi_i: U_i \rightarrow V$, where the (U_i)

are objects of \mathcal{S}_B^n ; for each i , φ_i is a homeomorphism from U_i to an open subset of V such that the diagram

$$\begin{array}{ccc} U_i & \xrightarrow{\varphi_i} & V \\ \pi_1 \downarrow & & \downarrow \pi \\ B & \xlongequal{\quad} & B \end{array}$$

commutes; finally, for all i and all j , the “change of chart” $\varphi_j^{-1} \circ \varphi_i$ is an isomorphism of \mathcal{S}_B from an open subset of U_i to an open subset of U_j .

The structure thus defined is that of a $(\mathcal{C}^0, \mathbb{C})$ -mixed space. If B is a \mathbb{C} -analytic space, and if the change of chart maps are all \mathbb{C} -analytic, then we have a \mathbb{C} -analytic mixed space. In this case, V itself is a \mathbb{C} -analytic space, and the fibres $V_x = \pi^{-1}(x)$ are \mathbb{C} -analytic sub-manifolds.

If B is a \mathcal{C}^∞ manifold (resp. \mathbb{R} -analytic, resp. \mathbb{C} -analytic), and if the change of chart maps are all \mathcal{C}^∞ (resp. ...), then we have a $(\mathcal{C}^\infty, \mathbb{C})$ -mixed manifold (resp. (\mathbb{R}, \mathbb{C}) , resp. (\mathbb{C}, \mathbb{C})). In this case, V itself is a manifold. Note that the notion of a (\mathbb{C}, \mathbb{C}) -mixed manifold, or a \mathbb{C} -analytic mixed manifold, reduces to simply having a \mathbb{C} -analytic manifold V endowed with a projection $\pi: V \rightarrow B$ onto another \mathbb{C} -analytic manifold such that π is of maximal rank at every point.¹

Let $\pi: V \rightarrow B$ and $\pi': V' \rightarrow B'$ be mixed spaces, and let $f_1: B \rightarrow B'$ be a continuous (resp. ...) map. Then a *morphism from V to V' over f_1* is a continuous map $f: V \rightarrow V'$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & V' \\ \pi \downarrow & & \downarrow \pi' \\ B & \xrightarrow{f_1} & B' \end{array}$$

commutes, and such that, for any charts $\varphi_i: U_i \rightarrow V$ and $\varphi'_j: U'_j \rightarrow V'$, the map $\varphi'_j \circ f \circ \varphi_i$ is a morphism of \mathcal{S}_{f_1} (resp. ...) from an open subset of U_i to U'_j .

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2. An equivalent definition

We now give another way of defining mixed spaces, equivalent to the above.

Given separated spaces B and V , along with a continuous map $\pi: V \rightarrow B$, the structure of a *pre-mixed space* consists of the structure of a \mathbb{C} -analytic manifold on each fibre $V_x = \pi^{-1}(x)$. Given pre-mixed spaces $\pi: V \rightarrow B$ and $\pi': V' \rightarrow B'$, along with a continuous map $f_1: B \rightarrow B'$, a *morphism of pre-mixed spaces over f_1* is a continuous map $f: V \rightarrow V'$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & V' \\ \pi \downarrow & & \downarrow \pi' \\ B & \xrightarrow{f_1} & B' \end{array}$$

commutes and induces a \mathbb{C} -analytic map on each fibre.

¹[Trans.] The more common modern nomenclature is to simply call such an object a family of complex manifolds.

A *mixed space* is a pre-mixed space $\pi: V \rightarrow B$ such that every point $y \in V$ admits a neighbourhood W in V that is isomorphic as a pre-mixed space to an open subset of $B \times \mathbb{C}^n$, via an isomorphism over the identity. The morphisms of mixed spaces are the same: mixed spaces form a *full subcategory*.

3. Deformations

A mixed space $\pi: V \rightarrow B$ is said to be *proper* if B is locally compact and the map π is proper (i.e. the inverse image of any compact subset is compact). If it is a mixed manifold, then we can show that it is a fibred manifold that is locally trivial with respect to the underlying \mathcal{C}^∞ structure, but the previous talk shows that, in general, any two fibres are not isomorphic as \mathbb{C} -analytic manifolds.

Definition. Let V_0 be a compact \mathbb{C} -analytic manifold, B a locally compact space, and $b_0 \in B$. Then a *\mathbb{C} -analytic deformation of V_0 over (B, b_0)* consists of a proper \mathbb{C} -analytic mixed space $\pi: V \rightarrow B$ along with an isomorphism of \mathbb{C} -analytic manifolds $i: V_0 \rightarrow \pi^{-1}(b_0)$.

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The goal of this seminar is the study, at least local, and an attempt at a classification of, \mathbb{C} -analytic deformations of a given compact \mathbb{C} -analytic manifold V_0 .

Definition. Let V_0 be a compact \mathbb{C} -analytic manifold. A *\mathbb{C} -analytic deformation $(\pi: V \rightarrow B, i: V_0 \rightarrow V)$ of V_0* is said to be *locally complete* if, for any other deformation $(\pi': V' \rightarrow B', i': V_0 \rightarrow V')$ of V_0 , there exists a neighbourhood B'_1 of b'_0 in B' , an analytic map $f_1: B'_1 \rightarrow B$ with $f_1(b'_0) = b_0$, and a morphism of \mathbb{C} -analytic mixed spaces $f: \pi'^{-1}(B'_1) \rightarrow V$ over f_1 such that $f \circ i' = i$. The deformation is said to be *locally universal* if furthermore the germ of f_1 at b'_0 is determined uniquely by this condition.

It seems that every compact \mathbb{C} -analytic manifold V_0 admits a locally complete \mathbb{C} -analytic deformation, and a locally universal one if the group of automorphisms of V_0 is discrete.

III. Vector fields

1. Study on models

Let B be a space, U an object of \mathcal{S}_B (i.e. an open subset of $B \times \mathbb{C}^n$), b_0 a point of B , and set $U_0 = \pi^{-1}(b_0)$.

A holomorphic field of tangent vectors on U_0 (i.e. a holomorphic map from U_0 to \mathbb{C}^n) is said to be a *vertical holomorphic field* on U_0 . A *vertical holomorphic field on U* is a continuous (resp. ...) map $\theta: U \rightarrow \mathbb{C}^n$ that induces a vertical holomorphic field on each fibre U_x . If $f: U \rightarrow U'$ is an isomorphism in \mathcal{S}_B , then the *transport $f_*\theta$ of θ by f* is defined by

$$f_*\theta(f(x, z)) = D_2f_{x,z} \cdot \theta(x, z)$$

where $D_2f_{x,z}$ is the linear map from \mathbb{C}^n to itself that is tangent to f_x at the point $z \in U_x$. This is again a vertical holomorphic field, since it follows from a Cauchy integral that the matrix $Df_{x,z}$ depends continuously on the pair (x, z) .

Now suppose that B is a \mathcal{C}^∞ manifold, just for simplicity, and let T_0 be the tangent space to B at b_0 . A field of tangent vectors to U defined on U_0 , i.e. a map $\omega: U_0 \rightarrow T_0 \times \mathbb{C}^n$, is said to be a *projectable holomorphic field* if $\omega(b_0, z) = (t_0, \theta(z))$ (where $t_0 \in T_0$ is a vector

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that does not depend on z , called the *projection* of the field ω) and $\theta(z)$ is a holomorphic vector field. If B is a \mathbb{C} -analytic space, possibly with a singularity at b_0 , then we give the same definition, but with T_0 then being the *Zariski* tangent space to B at b_0 , i.e. the dual of $\mathfrak{m}/\mathfrak{m}^2$, where \mathfrak{m} is the ideal of germs at b_0 of holomorphic functions on B that vanish at b_0 .

If $f : U \rightarrow U'$ is an isomorphism of $\mathcal{C}^\infty \mathcal{S}_B$ (resp. ...), then then transport $f_*\omega$ is defined by

$$f_*\omega(f(b_0, z)) = Df_{b_0, z}\omega(b_0, z)$$

where $Df_{b_0, z} : T_0 \times \mathbb{C}^n \rightarrow T_0 \times \mathbb{C}^n$ is now the linear map that is tangent to f at the point (b_0, z) . This is a projectable holomorphic field. Indeed, the matrix $Df_{b_0, z}$ can be written as

$$\begin{pmatrix} I & 0 \\ D_1f & D_2f \end{pmatrix}$$

and

$$\begin{aligned} D_1f &: T \rightarrow \mathbb{C}^n \\ D_2f &: \mathbb{C}^n \rightarrow \mathbb{C}^n \end{aligned}$$

both depend holomorphically on z (for D_1f , this follows from the fact that f_x is holomorphic for every x). By setting $f_*\omega(b_0, z') = (t_0, \theta'(z'))$, we have

$$\begin{aligned} \theta'(z') &= D_1f_{b_0, z}(t_0) + D_2f_{b_0, z}(\omega(z)) \\ &\text{if } z' = f_{b_0}(z) \end{aligned}$$

which shows that $f_*\omega$ is indeed a projectable holomorphic field.

A *projectable holomorphic field on U* is a \mathcal{C}^∞ field of vectors tangent to U that induces a projectable holomorphic field on each fibre.

2. Vector fields on a mixed manifold

Let $\pi : V \rightarrow B$ be a $(\mathcal{C}^\infty, \mathbb{C})$ -mixed manifold (resp. ..., resp. a \mathbb{C} -analytic mixed space). By transporting along the charts, we define the notions of

- vertical holomorphic fields on an open subset of a fibre;
- vertical holomorphic fields on a open subset of V ;
- projectable holomorphic fields on an open subset of a fibre; and
- projectable holomorphic fields on an open subset of V .

Let ξ be a \mathcal{C}^∞ vector field (resp. ...) on V . By integrating ξ , we obtain a \mathcal{C}^∞ map, denoted by e^ξ , from an open subset $W \subset \mathbb{R} \times V$ containing $\{0\} \times V$ (resp. \mathbb{C} -analytic map from an open subset $W \subset \mathbb{C} \times V$) to V , characterised by

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1. $e^\xi(t_1 + t_2, y) = e^\xi(t_1, e^\xi(t_2, y))$, with the left-hand side being defined whenever the right-hand side is; and
2. $\frac{\partial}{\partial t} e^\xi(t, y)|_{0, y} = \xi(y)$.

Note that W is a mixed manifold over $\mathbb{R} \times B$ (resp. a mixed space over $\mathbb{C} \times B$).

Proposition. For $e^\xi: W \rightarrow V$ to be a morphism of mixed spaces over the projection $\mathbb{R} \times B \rightarrow B$, it is necessary and sufficient for ξ to be a vertical holomorphic field. For $e^\xi: W \rightarrow V$ to be a morphism of mixed spaces over a map from an open subset of $\mathbb{R} \times B$ containing $\{0\} \times B$ to B , it is necessary and sufficient for ξ to be a projectable holomorphic field.

The proof is left to the reader.

IV. The Spencer–Kodaira map

Let $\pi: V \rightarrow B$ be a mixed manifold (resp. a \mathbb{C} -analytic mixed space), $b \in B$, and $V_0 = \pi^{-1}(b_0)$. Let T_0 be the tangent space to B at b_0 (resp. the Zariski tangent space). We introduce the following sheaves on V_0 :

- Θ_0 : the sheaf of germs of vertical holomorphic fields on V_0 ;
- Π_0 : the sheaf of germs of locally projectable holomorphic fields on V_0 ; and
- Λ_0 : the sheaf π^*T_0 , i.e. the sheaf of germs of locally constant maps from V_0 to T_0 .

We have an exact sequence of sheaves on V_0

$$0 \rightarrow \Theta_0 \rightarrow \Pi_0 \rightarrow \Lambda_0 \rightarrow 0$$

that gives rise to the long exact sequence in cohomology

$$\dots \rightarrow H^0(V_0; \Pi_0) \rightarrow H^0(V_0; \Lambda_0) \xrightarrow{\delta} H^1(V_0; \Theta_0) \rightarrow \dots$$

We also have a canonical map

$$\iota: T_0 \rightarrow H^0(V_0; \Lambda_0)$$

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that is injective if V_0 is non-empty, and surjective if V_0 is connected.

Definition. The *Spencer–Kodaira map* is the composition

$$\rho_0 = \delta \circ \iota: T_0 \rightarrow H^1(V_0; \Theta_0).$$

This map is an essential tool in the local study of deformations of \mathbb{C} -analytic varieties. Note that Θ_0 is exactly the sheaf of germs of holomorphic fields of tangent vectors to V_0 , and thus depends only on V_0 , while T_0 depends only on the base. Also, Θ_0 is a coherent analytic sheaf on V_0 , and, if V_0 is compact, then $H^1(V_0; \Theta_0)$ is a finite-dimensional vector space over \mathbb{C} [1]. We thus see that, in this case (which is the only case where we can say anything non-trivial), ρ_0 might be possible to calculate.

It is clear that, if the given mixed manifold is trivial (i.e. if $V = B \times V_0$, with π being the projection to B), then the map ρ_0 is zero. The next talk aims to show that, in a certain sense, ρ indicates the non-triviality of V in a neighbourhood of V_0 .

3. Regular deformations

I. The map $\tilde{\rho}$

All throughout this talk, B is a \mathcal{C}^∞ manifold (resp. \mathbb{R} -analytic, resp. \mathbb{C} -analytic); $\pi: V \rightarrow B$ denotes a proper mixed manifold; b_0 is a point of B ; and $V_0 = \pi^{-1}(b_0)$ is thus a compact \mathbb{C} -analytic manifold.

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Let $\tilde{\Theta}$ (resp. $\tilde{\Pi}$) be the sheaf of germs of vertical holomorphic (resp. locally projectable holomorphic) vector fields on V . The quotient sheaf $\tilde{\Lambda} = \tilde{\Pi}/\tilde{\Theta}$ is exactly the inverse image under π of the sheaf \tilde{T} of germs of \mathcal{C}^∞ fields (resp. . . .) of tangent vectors on B .

For every open subset U of B , set $V_U = \pi^{-1}(U)$. The exact sequence

$$0 \rightarrow \tilde{\Theta} \rightarrow \tilde{\Pi} \rightarrow \tilde{\Lambda} \rightarrow 0$$

of sheaves on V_U gives rise to a homomorphism

$$\tilde{\rho}_U: H^0(U; \tilde{T}) \xrightarrow{\pi_*} H^0(V_U; \tilde{\Lambda}) \xrightarrow{\delta} H^1(V_U; \tilde{\Theta}).$$

Let $R^1\pi_*\tilde{\Theta}$ be the sheaf on B defined by the presheaf $U \mapsto H^1(V_U; \tilde{\Theta})$. Then $\tilde{\rho}$ becomes a homomorphism of sheaves on B :

$$\tilde{\rho}: \tilde{T} \rightarrow R^1\pi_*\tilde{\Theta}.$$

In particular, we have a homomorphism

$$\tilde{\rho}_0: \tilde{T}_0 \rightarrow R^1\pi_*\tilde{\Theta} = H^1(V_0; \tilde{\Theta})$$

where \tilde{T}_0 is the vector space of germs at b_0 of fields of tangent vectors to B . Finally, we have a commutative diagram

| p. 3-02

$$\begin{array}{ccc} \tilde{T}_0 & \xrightarrow{\tilde{\rho}_0} & H^1(V_0; \tilde{\Theta}) \\ \varepsilon \downarrow & & \downarrow \varepsilon \\ T_0 & \xrightarrow{\rho_0} & H^1(V_0; \Theta_0) \end{array}$$

where ρ_0 is the Spencer–Kodaira map [?].

Theorem 1. *For the proper mixed manifold $\pi: V \rightarrow B$ to be locally trivial in a neighbourhood of the point $b_0 \in B$, it is necessary and sufficient for the map $\tilde{\rho}_0: \tilde{T}_0 \rightarrow H^1(V_0; \tilde{\Theta})$ to be zero.*

II. The regular case

For all $b \in B$, set $V_b = \pi^{-1}(b)$. Consider the family $\{H^1(V_b; \Theta_b)\}_{b \in B}$ of finite-dimensional \mathbb{C} -vector spaces, and, for all $b \in B$, the map

$$\varepsilon_b: H^1(V_b; \tilde{\Theta}) \rightarrow H^1(V_b; \Theta_b).$$

For every open subset $U \subset B$, we have a map

$$\tilde{\varepsilon}_U : H^1(V_U; \tilde{\Theta}) \rightarrow \prod_{b \in U} H^1(V_b; \Theta_b)$$

that defines, by varying U , a homomorphism from the sheaf $R^1\pi_*\tilde{\Theta}$ to the sheaf Φ on B defined by $\Phi(U) = \prod_{b \in U} H^1(V_b; \Theta_b)$.

Definition.

We say that the proper mixed manifold $\pi : V \rightarrow B$ is *regular* if

1. the dimension of $H^1(V_b; \Theta_b)$ does not depend on the point $b \in B$; and
2. we can endow $E = \bigcup_{b \in B} H^1(V_b; \Theta_b)$ with the structure of a \mathcal{C}^∞ vector bundle (resp. ...) such that $\tilde{\varepsilon}$ is an isomorphism from the sheaf $R^1\pi_*\tilde{\Theta}$ to the sheaf of germs of \mathcal{C}^∞ sections (resp. ...) of the bundle E .

In fact, Kodaira and Spencer have shown [7] that, by identifying the H^1 spaces with spaces of harmonic forms, condition (2) is a consequence of condition (1).

Then [Theorem 1](#) has the following corollary:

Proposition 1. *For the proper mixed manifold $\pi : V \rightarrow B$ to be locally trivial, it is necessary and sufficient for it to be regular and, for all $b \in B$, for the Spencer–Kodaira map*

$$\rho_b : T_b \rightarrow H^1(V_b; \Theta_b)$$

to be zero.

Indeed, since $\tilde{\varepsilon}$ is injective, this condition implies that the map

$$\tilde{\rho}_b : \tilde{T}_b \rightarrow H^1(V_b; \tilde{\Theta})$$

is zero for all b .

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At the end of this talk, we will construct a counter-example which shows that it is necessary to assume that the mixed manifold is regular.

III. An example of non-regular deformation: Hopf manifolds

1. Hopf manifolds

Let $n \geq 2$ be an integer, and let b be an $(n \times n)$ matrix with coefficients in \mathbb{C} , whose eigenvalues are all of modulus > 1 . The free group $L(b)$ generated by b acts freely on $\tilde{V} = \mathbb{C}^n \setminus \{0\}$, and the quotient space $\tilde{V}/L(b)$, which we call the *Hopf manifold defined by b* , is a compact \mathbb{C} -analytic manifold that is homeomorphic to $S^{2n-1} \times S^1$.

Note that V_b and $V_{b'}$ are isomorphic if and only if there exists some a such that $b' = aba^{-1}$ or $b' = ab^{-1}a^{-1}$ (cf. [Appendix](#)).

Let Θ be the sheaf of germs of holomorphic fields of tangent vectors on V_b .

Proposition 2. *We can identify $H^0(V_b; \Theta)$ with the vector space of matrices that commute with b , and $H^1(V_b; \Theta)$ has the same dimension as this vector space.*

Proof. If X is a vector field on an open subset $U \subset \tilde{V}$, then $b_*(X)$ is the vector field on the open subset $b(U)$ given by transporting via b , i.e. $b_*X(u) = bX(b^{-1}u)$. Let $\mathcal{U} = \{U_i\}$ be a cover of V by simply connected Stein open subsets; for all i , set $\tilde{U}_i = \chi^{-1}\{U_i\}$, where χ is the canonical map from \tilde{V} to V_b . The cover $\tilde{\mathcal{U}} = \{\tilde{U}_i\}$ of \tilde{V} consists of Stein open subsets that are invariant under b (not necessarily connected, but this doesn't matter). Then b_* defines a map, again denoted by b_* , from the group of cochains $C^*(\tilde{V}, \tilde{\mathcal{U}}; \Theta)$ to itself.

Lemma 1. *We have the exact sequence*

$$0 \rightarrow C^*(V_b, \mathcal{U}; \Theta) \xrightarrow{\chi^*} C^*(\tilde{V}, \tilde{\mathcal{U}}; \Theta) \xrightarrow{1-b_*} C^*(\tilde{V}, \tilde{\mathcal{U}}; \Theta) \rightarrow 0.$$

Proof. The only thing that we need to verify is that the map $1 - b_*$ is surjective. For all (i_0, \dots, i_q) , let U'_{i_0, \dots, i_q} be an open subset of \tilde{V} such that

$$\chi: U'_{i_0, \dots, i_q} \rightarrow U_{i_0, \dots, i_q}$$

is a homeomorphism. The $\tilde{U}_{i_0, \dots, i_q}$ is a disjoint union of the $b_*^p U'_{i_0, \dots, i_q}$, where $p \in \mathbb{Z}$, and every $\gamma \in C^q(\tilde{V}, \tilde{\mathcal{U}}; \Theta)$ can be written in the form $\gamma = \gamma_1 - \gamma_2$, with $\gamma_1 = 0$ on $b^p(U'_{i_0, \dots, i_q})$ for $p < 0$, and $\gamma_2 = 0$ for $p \geq 0$. Set

$$\beta = \sum_{p \geq 0} b_*^p \gamma_1 + \sum_{p < 0} b_*^p \gamma_2$$

(which is a locally finite sum). Then $\beta - b_*\beta = \gamma$, whence [Lemma 1](#). □

Now, to finish the proof of [Proposition 2](#). From [Lemma 1](#), we have the following exact sequence:

$$0 \rightarrow H^0(V_b; \Theta) \xrightarrow{\chi^*} H^0(\tilde{V}; \Theta) \xrightarrow{1-b_*} H^0(\tilde{V}; \Theta) \xrightarrow{\delta_*} H^1(V_b; \Theta) \xrightarrow{\chi^*} H^1(\tilde{V}; \Theta) \xrightarrow{1-b_*} H^1(\tilde{V}; \Theta).$$

We can show that

$$\chi^*: H^1(V_b; \Theta) \rightarrow H^1(\tilde{V}; \Theta)$$

is zero: if $n > 2$, it is evident, since $H^1(\tilde{V}; \Theta) = 0$; if $n = 2$, then a direct calculation on the cochains of a cover of \tilde{V} by two Stein open subsets shows that

$$1 - b_*: H^1(\tilde{V}; \Theta) \rightarrow H^1(\tilde{V}; \Theta)$$

is bijective.

Now $H^0(\tilde{V}; \Theta)$ is the space of holomorphic vector fields on \tilde{V} , but such a field extends to a holomorphic vector field on \mathbb{C}^n , and $H^0(\tilde{V}, \Theta) = L \oplus M$, where L is the space of fields of linear vectors, and M is the space of fields of second-order vectors at 0. The subspaces L and M are invariant under b_* , and $1 - b_*: M \rightarrow M$ is an isomorphism. Then [Proposition 2](#) follows from remarking that, if an element of L is represented by a matrix a , then $b_*a = bab^{-1}$. □

2. Mixed manifolds whose fibres are Hopf manifolds

Let B be the set of all $(n \times n)$ matrices with coefficients in \mathbb{C} with eigenvalues all of modulus > 1 . This is an open subset of \mathbb{C}^{n^2} . Let α be the transformation from $B \times \tilde{V}$ to itself defined by $\alpha(b, x) = (b, b(x))$. The free group $L(\alpha)$ generated by α acts linearly on $B \times \tilde{V}$, and the quotient $V = B \times \tilde{V}/L(\alpha)$ is a \mathbb{C} -analytic manifold. By endowing it with the projection $\pi: V \rightarrow B$ induced by the projection $\pi_1: B \times \tilde{V} \rightarrow B$ after passing to the quotient, we obtain a \mathbb{C} -analytic mixed manifold that is proper, but not regular. Indeed, condition 1 of the definition of regular mixed manifolds is not satisfied: for example, for $n = 2$, the dimension of $H^1(V_b; \Theta)$ is 4 if b is a scalar matrix, but 2 in all other cases. | p. 3-06

Note that the dimension of $H^1(V_b; \Theta_b)$ is an upper semi-continuous function of b , and that the set of b such that $\dim H^1(V_b; \Theta_b) \geq k$ is a closed analytic subspace of B . This is a general result, that we hope to be able to prove in a later talk of this seminar.

3. Calculation of ρ

We have $T_b = \text{Hom}(\mathbb{C}^n, \mathbb{C}^n) = L \subset H^0(\tilde{V}; \Theta)$, and we defined, to prove [Proposition 2](#), a surjective map $\delta_*: L \rightarrow H^1(V_b; \Theta)$.

Proposition 3. *The Spencer–Kodaira map ρ is given, for the mixed manifold studied in this section, by*

$$\rho(a) = \delta_*(ab^{-1}).$$

In particular, it is surjective, and its kernel is the space of matrices of the form $[\ell, b]$ for $\ell \in L$.

Proof. Let $a \in T_b = L$. Let $\{U_i\}$ be a cover of V_b by simply connected Stein open subsets, and, for each i , let U'_i be a connected component of \tilde{U}_i .

Let η'_i be the projectable holomorphic field on U'_i defined by $\eta'_i(x) = (a, 0)$; let $\tilde{\eta}_i$ be the projectable holomorphic field on \tilde{U}_i defined by $\tilde{\eta}_i = \alpha_*^k \eta'_i$ on $b^k(U'_i)$; and let η_i be the projectable holomorphic field on U_i corresponding to $\tilde{\eta}_i$. By definition, $\rho(a)$ is the cohomology class of the cochain $\{\theta_{ij}\}$, where $\theta_{ij} = \eta_j - \eta_i$ is a vertical holomorphic field on U_{ij} . | p. 3-07

Set $\tilde{\eta}_i(x) = (a, \beta_i(x))$. Then $\beta \in C^0(\tilde{V}; \Theta)$, and we have $(1 - b_*)\beta = ab^{-1} \in L \subset H^0(\tilde{V}; \Theta)$. Indeed, $\alpha_*\eta = \eta$, $\alpha_*\eta_i(b_{-1}x) = \eta_i(x)$, and

$$\alpha_*(a, \beta(b^{-1}x)) = (a, \beta(x)),$$

whence

$$ab^{-1}x + b \cdot \beta(b^{-1}x) = \beta(x).$$

We thus deduce that $\theta = \delta_*(ab^{-1})$, which proves [Proposition 3](#). □

4. A counter-example

Take $n = 2$, and $\sigma \in \mathbb{C}$ such that $|\sigma| > 1$. Let $B' \subset B$ be the set of matrices of the form

$$\begin{pmatrix} \sigma & t \\ 0 & \sigma \end{pmatrix}$$

where $t \in \mathbb{C}$, and let $V' = \pi^{-1}(B')$ be the mixed manifold induced by V over V' ; now B' is a line, and its tangent space T'_b at b is generated, for all b , by $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. It follows from [Proposition 3](#) that the Spencer–Kodaira map

$$\rho': T'_b(B') \rightarrow H^1(V_b; \Theta)$$

is zero if and only if

$$b \neq b_0 = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

since, if $b \neq b_0$, then $a = [\ell, b]$, where $\ell = \begin{pmatrix} t^{-1} & 0 \\ 0 & 0 \end{pmatrix}$; and if $b = b_0$, then ρ' is injective.

We can also see that V' is trivial on $B' \setminus \{b_0\}$.

Let $\varphi: \mathbb{C} \rightarrow B' \subset B$ be the map defined by

$$\varphi(t) = \begin{pmatrix} \sigma & t^2 \\ 0 & \sigma \end{pmatrix}$$

and let V^φ be the mixed manifold given by the inverse image of V under φ . The Spencer–Kodaira map ρ'_t from \mathbb{C} to $H^1(V_{\varphi(t)}; \Theta)$ is the composition

| p. 3-08

$$\rho'_{\varphi(t)} \circ D\varphi: \mathbb{C} \rightarrow T'_{\varphi(t)} \rightarrow H^1(V_{\varphi(t)}; \Theta),$$

and this is zero for all t , since, if $t \neq 0$, then $\rho'_{\varphi(t)}$ is zero; and, if $t = 0$, then $D\varphi$ is zero.

However, the mixed manifold V^φ is not locally trivial, since V_0^φ is not isomorphic to V_t^φ for $t \neq 0$.

5. Question (K. Srinivasacharyulu)

We know that the Hopf manifolds are non-Kähler, and thus non-algebraic. For $n = 2$, the manifold V_b admits non-constant meromorphic functions if and only if b can be diagonalised with eigenvalues σ_1 and σ_2 satisfying $\sigma_1^p = \sigma_2^q$ for some integers p and q (and there is then the function $x_1^p x_2^{-q}$). The set of b satisfying this property is neither open nor closed, but it is a countable union of closed analytic subspaces. An analogous phenomenon arises for deformations of complex tori. Is this result general?

Appendix

With the notation of [§III.1](#), let $f: V_b \rightarrow V_{b'}$ be an isomorphism of \mathbb{C} -analytic manifolds. This lifts to an isomorphism of universal coverings

$$\tilde{f}: \mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{C}^n \setminus \{0\}.$$

By Hartog, \tilde{f} extends to an isomorphism $g: \mathbb{C}^n \rightarrow \mathbb{C}^n$. We necessarily have

$$g(bz) = (b')^k g(z) \tag{*}$$

where $z \in \mathbb{C}^n$, and k is an integer; the same property, applied to the inverse map of g , shows that $k = \pm 1$. Let a be the linear map that is tangent to g at the origin; the identity (*) then gives

$$ab = (b')^k a$$

$$k = \pm 1$$

whence

$$b' = aba^{-1} \quad \text{or} \quad b' = ab^{-1}a^{-1}.$$

4. The primary obstruction to deformation

Introduction

| p. 4-01

Let V_0 be a compact complex-analytic manifold, and let Θ be the sheaf of germs of holomorphic fields of tangent vectors. We ask the following question: given an element $a \in H^1(V_0, \Theta)$, does there exist a deformation of V_0 , with a non-singular base (i.e. a fibred mixed manifold $\pi: V \rightarrow B$, with $b_0 \in B$, along with an isomorphism $V_0 \xrightarrow{\cong} \pi^{-1}(b_0)$), such that a is the image, under the map ρ defined in [Talk no. 2], of a vector v that is tangent to B at b_0 ? An element $a \in H^1(V_0, \Theta)$ for which the answer is positive is called a *deformation vector*. We will give a necessary condition for a to be a deformation vector; this condition is written $[a \smile a] = 0$. We will then give an example where this condition is not satisfied.

I. Exact sequences of sheaves of algebras

Let K be a commutative ring, and let Φ , Φ_1 , and Φ_2 be sheaves of K -modules on some space X , and suppose that we have some given homomorphism $\Phi_1 \otimes \Phi_2 \rightarrow \Phi$, written as a product. We define, for any cover \mathcal{U} of X , the *cup product*

$$\smile: C^p(X, \mathcal{U}; \Phi_1) \otimes C^q(X, \mathcal{U}; \Phi_2) \rightarrow C^{p+q}(X, \mathcal{U}; \Phi)$$

by the formula

$$(\alpha \smile \beta)_{i_0, \dots, i_{p+q}} = \alpha_{i_0, \dots, i_p} \cdot \beta_{i_p, \dots, i_{p+q}}.$$

We have the relation

$$d(\alpha \smile \beta) = d\alpha \smile \beta + (-1)^p \alpha \smile d\beta.$$

This induces a cup product on the cohomology of the cover \mathcal{U} , and, by passing to the inductive limit over open covers, a cup product

$$\smile: H^p(X; \Phi_1) \otimes H^q(X; \Phi_2) \rightarrow H^{p+q}(X; \Phi).$$

| p. 4-02

Definition. A *sheaf of algebras* on X is a sheaf of modules Φ on X endowed with a product $\Phi \otimes \Phi \rightarrow \Phi$ (which we do not assume to be either commutative nor associative).

If $f: \Phi \rightarrow \Psi$ is a homomorphism of sheaves of algebras, then the kernel Φ' of f is a sheaf of two-sided ideals of Φ , i.e. we have products $\Phi' \otimes \Phi \rightarrow \Phi'$ and $\Phi \otimes \Phi' \rightarrow \Phi'$ such that the two diagrams

$$\begin{array}{ccc} \Phi' \otimes \Phi & \longrightarrow & \Phi' \\ \downarrow & & \downarrow \\ \Phi \otimes \Phi & \longrightarrow & \Phi \end{array} \quad \begin{array}{ccc} \Phi \otimes \Phi' & \longrightarrow & \Phi' \\ \downarrow & & \downarrow \\ \Phi \otimes \Phi & \longrightarrow & \Phi \end{array}$$

both commute.

Proposition 1. Let $0 \rightarrow \Phi' \rightarrow \Phi \rightarrow \Phi'' \rightarrow 0$ be an exact sequence of sheaves of algebras on X ; let $a \in H^p(X; \Phi'')$. Then $\delta a \in H^{p+1}(X; \Phi')$, and, for any class $b \in H^q(X; \Phi')$, we have $\delta a \smile b = 0$.

Proof. Let \mathcal{U} be a cover of X such that a and b are represented by cocycles α and β (respectively), and such that α lifts to a cochain $\eta \in C^p(X, \mathcal{U}; \Phi)$. Then $\delta\eta$ is a cocycle in $C^{p+1}(X, \mathcal{U}; \Phi')$ whose class in $H^{p+1}(X; \Phi')$ is, by definition, δa , and $\delta a \smile b$ is the class of $\delta\eta \smile \beta$. But $\delta(\eta \smile \beta) = \delta\eta \smile \beta$, and $\eta \smile \beta$ is a cochain in $C^{p+q}(X, \mathcal{U}; \Phi')$, since Φ' is a sheaf of ideals. So the cocycle $\delta\eta \smile \beta$ is cohomologous to 0 in $H^{p+q+1}(X; \Phi')$, which proves the proposition. \square

II. The primary obstruction

Let V_0 be a complex-analytic manifold, and Θ_0 the sheaf of germs of holomorphic fields of tangent vectors. Then Θ_0 is a sheaf of Lie algebras, and, if $a, b \in H^*(V_0, \Theta_0)$, then we denote by $[a \smile b]$ the cup product defined by the bracket $[-, -]: \Theta_0 \otimes \Theta_0 \rightarrow \Theta_0$. It satisfies

$$[b \smile a] = (-1)^{pq+1}[a \smile b]$$

for $a \in H^p(V_0, \Theta_0)$ and $b \in H^q(V_0, \Theta_0)$.

| p. 4-03

Theorem 1. Let $\pi: V \rightarrow B$ be a mixed manifold, b_0 a point of B , $V_0 = \pi^{-1}(b_0)$, and let $\rho_0: T_0 \rightarrow H^1(V_0, \Theta_0)$ be Spencer–Kodaira map. Then, if u and v are tangent vectors of B at b_0 , we have

$$[\rho_0(u) \smile \rho_0(v)] = 0.$$

Corollary. Let V_0 be a complex-analytic manifold, and Θ the sheaf of germs of holomorphic fields of tangent vectors of V_0 . If $a \in H^1(V_0, \Theta)$ is a deformation vector, then

$$[a \smile a] = 0.$$

Proof. (Proof of the Corollary). This is simply a particular case of [Theorem 1](#); note that $[a \smile b]$ is a symmetric bilinear map from $H^1 \otimes H^1$ to H^2 , and that we are in characteristic $0 \neq 2$. \square

Proof. (Proof of Theorem 1). Consider the following sheaves on V_0 :

- Θ_0 : the sheaf of germs of vertical holomorphic fields on V_0 ;
- $\tilde{\Theta}_0$: the sheaf of germs of vertical holomorphic fields on V ;
- Π_0 : the sheaf of germs of locally projectable holomorphic fields on V_0 ;
- $\tilde{\Pi}_0$: the sheaf of germs of locally projectable holomorphic fields on V ;
- Λ_0 : the sheaf $\pi^* T_0$, where T_0 is the tangent space of B at b_0 ; and
- $\tilde{\Lambda}_0$: the sheaf $\pi^* \tilde{T}_0$, where \tilde{T}_0 is the space of germs at b_0 of fields on B of tangent vectors of B .

We have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{\Theta}_0 & \longrightarrow & \tilde{\Pi}_0 & \longrightarrow & \tilde{\Lambda}_0 & \longrightarrow & 0 \\ & & \varepsilon \downarrow & & \varepsilon \downarrow & & \varepsilon \downarrow & & \\ 0 & \longrightarrow & \Theta_0 & \longrightarrow & \Pi_0 & \longrightarrow & \Lambda_0 & \longrightarrow & 0 \end{array}$$

whence we obtain the following commutative diagram:

| p. 4-04

$$\begin{array}{ccc} \tilde{T}_0 & \xrightarrow{\tilde{\rho}} & H^1(V_0; \tilde{\Theta}) \\ \varepsilon \downarrow & & \downarrow \varepsilon \\ T_0 & \xrightarrow{\rho} & H^1(V_0; \Theta_0) \end{array}$$

Let $u, v \in T_0$ be fixed tangent vectors of B at b_0 . We can always find vector fields \tilde{u} and \tilde{v} on B that take the values u and v (respectively) at b_0 ; $\varepsilon(\tilde{u}) = u$ and $\varepsilon(\tilde{v}) = v$. The exact sequence

$$0 \rightarrow \tilde{\Theta}_0 \rightarrow \tilde{\Pi}_0 \rightarrow \tilde{\Lambda}_0 \rightarrow 0$$

is a sequence of homomorphisms of sheaves of Lie algebras, and so

$$[\tilde{\rho}(\tilde{u}) \smile \tilde{\rho}(\tilde{v})] = 0$$

by [Proposition 1](#). But $\varepsilon: \tilde{\Theta}_0 \rightarrow \Theta_0$ is also a homomorphism of sheaves of Lie algebras, and the diagram

$$\begin{array}{ccc} H^1(V_0, \tilde{\Theta}_0) \otimes H^1(V_0, \tilde{\Theta}_0) & \xrightarrow{[-\smile-]} & H^2(V_0, \tilde{\Theta}_0) \\ \varepsilon \otimes \varepsilon \downarrow & & \downarrow \varepsilon \\ H^1(V_0, \Theta_0) \otimes H^1(V_0, \Theta_0) & \xrightarrow{[-\smile-]} & H^2(V_0, \Theta_0) \end{array}$$

commutes. We thus deduce that $[\rho(u) \smile \rho(v)] = 0$.

□

| p. 4-05

Remarks.

—

1. We make essential use of the fact that $\varepsilon: \tilde{T}_0 \rightarrow T_0$ is surjective, and thus of the fact that B has no singularities.

2. We actually have $[\rho(u) \smile b] = 0$ for all $u \in T_0$, for any class $b \in H^1(V_0, \Theta_0)$ that is in the image of $H^1(V_0, \tilde{\Theta}_0)$ under ϵ . In particular, for an element $a \in H^1(V_0, \Theta_0)$ to be a regular deformation vector (in the sense of [Talk no. 3]), it is necessary and sufficient for $[a \smile b] = 0$ for all $b \in H^1(V_0, \Theta_0)$.

If V_0 is a compact complex-analytic manifold, and $a \in H^1(V_0, \Theta)$, then we call $[a \smile a] \in H^2(V_0, \Theta)$ the *primary obstruction* to the deformation of V_0 along a . For a to be a deformation vector, it is necessary that this primary obstruction be zero; but it is not sufficient: we can define a sequence of set-theoretic maps ω_n , called *obstructions*, with $\omega_1: H^1(V_0, \Theta) \rightarrow H^2(V_0, \Theta)$ given by $\omega_1(a) = [a \smile a]$, and with ω_{k+1} defined on the subset of $H^1(V_0, \Theta)$ where ω_k vanishes, with values in varying quotients² of $H^2(V_0, \Theta)$, and a necessary condition for a to be a deformation vector is that all the $\omega_k(a)$ be defined and real. I do not know if *this* condition is sufficient. Kodaira, Spencer, and Nijenhuis [5] have shown that, if $H^2(V_0, \Theta) = 0$, then every element of $H^1(V_0, \Theta)$ is a deformation vector. In this case, we even have a locally universal deformation whose base is a manifold, and ρ is an isomorphism from the tangent space of this manifold to $H^1(V_0, \Theta)$

III. An example of obstruction

1. The manifold V_0

Let $X = E/\Gamma$ be a 2-dimensional complex torus, i.e. $E \cong \mathbb{C}^2$ and $\Gamma \cong \mathbb{Z}^4$, and let D be the projective line $\mathbb{P}^1\mathbb{C}$. Set $V_0 = X \times D$. The sheaf Θ of holomorphic fields of tangent vectors of V_0 is the direct sum of the sheaves of Lie algebras Θ_1 and Θ_2 , where

$$\begin{aligned}\Theta_1 &= \mathcal{O} \otimes_{\mathcal{O}_X} \pi_1^* \Theta_X \\ \Theta_2 &= \mathcal{O} \otimes_{\mathcal{O}_D} \pi_2^* \Theta_D\end{aligned}$$

where $\pi_1: V_0 \rightarrow X$ and $\pi_2: V_0 \rightarrow D$ are the projections, \mathcal{O} , \mathcal{O}_X , and \mathcal{O}_D are the structure sheaves (sheaves of local rings), and Θ_X and Θ_D are the sheaves of germs of holomorphic fields of tangent vectors of X and D (respectively). We are mostly interested in Θ_2 . Also, $H^1(V_0, \Theta_2)$ is given by the Künneth exact sequence: | p. 4-06

$$0 \rightarrow H^0(X, \mathcal{O}_X) \otimes H^1(D, \Theta_D) \rightarrow H^1(V_0, \Theta_2) \rightarrow H^1(X, \mathcal{O}_X) \otimes H^0(D, \Theta_D) \rightarrow 0.$$

But we know that $H^0(D, \Theta_D)$ is the Lie algebra \mathfrak{a} of the group

$$A = \mathrm{GL}(2, \mathbb{C})/\mathbb{C}^* = \mathrm{SL}(2, \mathbb{C})/\{\pm 1\}$$

of automorphisms of D , and that $H^1(D, \Theta_D) = 0$, as we can easily see by taking a cover of D by two open subsets. We have already seen (in [Talk no. 1]) that, if $X = E/\Gamma$, then $H^1(X, \mathcal{O}) = \mathrm{Hom}(\Gamma, \mathbb{C})/\mathrm{Hom}_{\mathbb{C}}(E, \mathbb{C})$ is of dimension 2. So $H^1(V_0, \Theta_2) = H^1(X, \mathcal{O}) \otimes \mathfrak{a}$ is of dimension 6. The cup product

$$H^1(V_0, \Theta_2) \otimes H^1(V_0, \Theta_2) \rightarrow H^2(V_0, \Theta_2)$$

²See the [Appendix](#).

is given by the formula

$$[(\gamma \otimes \alpha) \smile (\gamma' \otimes \alpha')] = (\gamma \smile \gamma') \otimes [\alpha, \alpha'].$$

The cone of elements $\varphi \in H^1(V_0, \Theta_2)$ such that $[\varphi \smile \varphi] = 0$ can be identified with the cone of rank 1 tensors in $H^1(X, \mathcal{O}) \otimes \mathfrak{a}$. Indeed, if $\varphi = \gamma \otimes \alpha$, then

$$[\varphi \smile \varphi] = (\gamma \smile \gamma) \otimes [\alpha, \alpha] = 0 \otimes 0 = 0$$

and, if φ is not a simple tensor, then we have

$$\varphi = \gamma \otimes \alpha + \gamma' \otimes \alpha'$$

with γ and γ' independent, and α and α' independent, so

$$[\varphi \smile \varphi] = 2(\gamma \smile \gamma') \otimes [\alpha, \alpha'] \neq 0.$$

2. The mixed space V

In this example, every element of $H^1(V_0, \Theta_2)$ whose primary obstruction is zero is a deformation vector. More precisely:

| p. 2-07

Proposition 2.

There exists a mixed space $\pi: V \rightarrow B$ and a point $b_0 \in B$ such that

1. $\pi^{-1}(b_0) = V_0$ (the manifold defined in §III.1);
2. there exists an isomorphism σ from a \mathbb{C} -analytic space B to the cone of elements $\varphi \in H^1(V_0, \Theta_2)$ such that $[\varphi \smile \varphi] = 0$; and
3. for every subspace B' of B that has no singularities at b_0 , the Spencer-Kodaira map ρ from the tangent space of B' at b_0 to $H^1(V_0, \Theta)$ agrees with $\sigma: B' \rightarrow H^1(V_0, \Theta_2)$.

Let H be the analytic space of homomorphisms from Γ to \mathfrak{a} whose images are contained in a vector subspace of \mathfrak{a} that is 1-dimensional over \mathbb{C} (i.e. (4×2) matrices of rank 1 with coefficients in \mathbb{C}). For every $h \in H$, $e \circ h$ is a homomorphism from Γ to A , where $e: \mathfrak{a} \rightarrow A$ denotes the exponential map, and we construct a manifold V_h that is fibred over X with fibre D as follows: V_h is the quotient of $E \times D$ by the equivalence relation defined by Γ acting via

$$\gamma \star (x, y) = (x + \gamma, ((e \circ h)(\gamma)) \cdot y).$$

These manifolds are the fibres of a mixed space $W \rightarrow H$, where W is the quotient of $H \times E \times D$ by the equivalence relation defined by Γ acting via

$$\gamma \star (h, x, y) = (h, x + \gamma, (e \circ h(\gamma)) \cdot y).$$

We now place the following equivalence relation on H : we have $h' \sim h$ if and only if $(h' - h)$ extends to an \mathbb{C} -linear map $f: E \rightarrow \mathfrak{a}$. Note that, if $h'(\Gamma)$ and $h(\Gamma)$ are contained in the same subspace L of \mathfrak{a} of dimension 1 over \mathbb{C} (or if $h' \sim h$), then we also have $f(E) \subset L$ (or $h \sim 0$ and $h' \sim 0$). In both cases, V_h and $V_{h'}$ are isomorphic, and we have an isomorphism $i_{h',h}: V_h \rightarrow V_{h'}$ defined by

$$i_{h',h}(x, y) = (x, e \circ f(x) \cdot y)$$

(in the first case), or

$$i_{h',h} = i_{h',0} \circ i_{0,h}$$

(in the second case). If h, h' , and h'' are in the same class, then we have $i_{h''h} = i_{h''h'} \circ i_{h'h}$, and we can place on W the equivalence relation | p. 4-08

$$(h', z') \sim (h, z) \iff h' \sim h \text{ or } z' = i_{h'h}z$$

for $h, h' \in H$, $z \in V_h$, and $z' \in V_{h'}$.

Let B and V be the quotients of H and W (respectively) by these equivalence relations. We have a projection $V \rightarrow B$. To show that the structures of a \mathbb{C} -analytic space on H and W induce structures of a \mathbb{C} -analytic space on their quotients B and V , it suffices to remark that we can lift B to a analytic subspace of H : let, for example, $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ be a basis of Γ such that (γ_1, γ_2) is a basis of E over \mathbb{C} ; then each class $b \in B$ contains exactly one element $h \in H$ such that

$$h(\gamma_1) = h(\gamma_2) = 0.$$

3. Calculating ρ_0

Let T be the Zariski tangent space of B at b_0 , i.e. the dual of $\mathcal{J}/\mathcal{J}^2$, where \mathcal{J} is the ideal of germs at b_0 of analytic functions on B that are zero at b_0 . Then T_0 can be identified with $\text{Hom}(\Gamma, a)/\text{Hom}_{\mathbb{C}}(E, a)$. Also,

$$\begin{aligned} H^1(V_0, \Theta) &= H^1(V_0; \Theta_1) \oplus H^1(V_0; \Theta_2) \\ &= (H^1(X; \mathcal{O}) \otimes E) \oplus (H^1(X; \mathcal{O}) \otimes a), \end{aligned}$$

and the second term of this term can be identified with the quotient $\text{Hom}(\Gamma, a)/\text{Hom}_{\mathbb{C}}(E, a)$. We are going to show that the map $\rho_0: T_0 \rightarrow H^1(V_0; \Theta)$ is exactly the canonical injection defined by these identifications.

Let $u \in T_0 = \text{Hom}(\Gamma, a)/\text{Hom}(E, a)$ be the class of an element $h \in \text{Hom}(\Gamma, a)$, which we suppose to be of rank 1. Then we can write h in the form $\eta \otimes \sigma$, where $\eta \in \text{Hom}(\Gamma, \mathbb{C})$, $\sigma \in a$, and we can consider h as a tangent vector to H at 0. Let \bar{h} be the field of tangent vectors to $H \times E \times D$ at $0 \times E \times D$ that projects onto h , and thus whose components over $E \times D$ are zero. Let (U_i) be a cover of $X = E/\Gamma$ by simply connected open subsets, and choose, for each i , a component \tilde{U}_i of the inverse image of U_i in E . We will denote by v_i the image over $U_i \times D$ of the field $\bar{h}|_{\tilde{U}_i \times D}$. This is a projectable holomorphic field on $0 \times U_i \times D$ of tangent vectors of $H \times U_i \times D$, and we set $w_{ij} = v_j - v_i$, so that w_{ij} is a vertical holomorphic field on $U_{ij} \times D$, and these fields form a cocycle whose cohomology class will be, by definition, $\rho_0(u)$. | p. 4-09

Let $x \in U_{ij}$, and let \tilde{x}_i and \tilde{x}_j be its inverse image in \tilde{U}_i and \tilde{U}_j (respectively). We have that $\tilde{x}_j = \tilde{x}_i + \gamma_{ij}(x)$, where $\gamma_{ij}(x) \in \Gamma$, and

$$w_{ij}(x) = \bar{h}(\tilde{x}_j) - [\gamma_{ij}(x)]_* (\bar{h}(\tilde{x}_i)) = -h(\gamma_{ij}(x)) \in a.$$

Now w_{ij} is a vector field on D , and so

$$(w_{ij}) \in Z^1(V_0, (U_i \times D); \Theta_2),$$

and w_{ij} is of the form $\zeta \otimes \alpha$, where $\zeta \in Z^1(V_0, (U_i \times D); \mathcal{O})$ is the cocycle defined by $\zeta_{ij}(x) = -\eta(\gamma_{ij}(x))$. This is a cocycle whose cohomology class is (up to a sign) the element of $H^1(V_0, \mathcal{O})$ that is identified with the class η in $\text{Hom}(\Gamma, \mathbb{C})/\text{Hom}_{\mathbb{C}}(E, \mathbb{C})$. QED.

Appendix: Higher obstructions

I. Definition of obstructions

1. The sheaf of germs of vertical automorphisms

Let V_0 be a \mathbb{C} -analytic manifold, which we assume to be compact, and B a \mathbb{C} -analytic space, and let $b_0 \in B$. We are going to define a sheaf Γ of non-abelian groups on V_0 . For every open subset U of V_0 , consider the isomorphisms of analytic varieties $\gamma: W \rightarrow W'$, where W and W' are open subsets of $B \times V_0$ that contain $\{b_0\} \times U$, such that the following conditions are satisfied: | p. 4-10

1. $\pi_1 \gamma = \pi_1$ is the projection $B \times V_0$ to B ;
2. γ is the identity on $\{b_0\} \times U$.

Then $\Gamma(U)$ consists of equivalence classes of these isomorphisms, where we identify γ_1 with γ_2 if they agree on a neighbourhood of $\{b_0\} \times U$.

It is clear that $\Gamma(U)$ is a group under composition of isomorphisms, and that the $\Gamma(U)$ form a sheaf Γ of non-abelian groups.

Proposition 1. *We can identify $H^1(V_0, \Gamma)$ with the set of classes of deformation germs of V_0 over (B, b_0) .*

Recall that a deformation germ of V_0 over (B, b_0) is a deformation of V_0 over a neighbourhood of b_0 in B , and that two such deformations $(B', b_0, V', \pi', \iota')$ and $(B'', b_0, V'', \pi'', \iota'')$ are locally equivalent if there exists a neighbourhood W' of $(\pi')^{-1}(b_0)$ in V' , a neighbourhood W'' of $(\pi'')^{-1}(b_0)$ in V'' , and an isomorphism φ from W' to W'' such that the diagram

$$\begin{array}{ccc}
 V_0 & \xlongequal{\quad} & V_0 \\
 \downarrow & & \downarrow \\
 W' & \xrightarrow{\quad \varphi \quad} & W'' \\
 \pi' \downarrow & & \downarrow \pi'' \\
 B & \xlongequal{\quad} & B
 \end{array}$$

commutes. | p. 4-11

Proof. (Proof of Proposition 1). Let (B', b_0, V, π, ι) be a deformation of V_0 over a neighbourhood V' of b_0 in B . Then we can find a cover $\{U_i\}$ of V_0 and a cover $\{W_i\}$ of a neighbourhood of $\iota(V_0)$ in V , along with isomorphisms $\{h_i\}$, where h_i is an isomorphism from a neighbourhood of $\{b_0\} \times U_i$ in $B \times V_0$ to W_i that agrees with ι on $\{b_0\} \times U_i$, and such that $\pi \circ h_i = \pi_1$.

Set $\gamma_{ij} = h_i^{-1} \circ h_j$. We can show that the γ_{ij} define an element of $\Gamma(U_i \cap U_j)$, and that $\gamma_{ij} \circ \gamma_{jk} = \gamma_{ik}$. The γ_{ij} thus form a cocycle $\gamma \in Z^1(V_0, \{U_i\}; \Gamma)$. Such a cocycle is said to be

associated to the deformation. It will still be associated to the deformation if pass to a finer cover. Let (B', b_0, V', π', t') be a deformation that is locally equivalent to the first, and let γ' be a cocycle associated to this deformation. We can suppose, by refining the covers if necessary, that the cocycles γ and γ' are defined with respect to the same cover $\{U_i\}$ of V_0 . Let f be an isomorphism from a neighbourhood of $t(V_0)$ in V to a neighbourhood of $t'(V_0)$ in V' . Set $f_i = (h'_i)^{-1} \circ f \circ h_i$. Then $f_i \in \Gamma(U_i)$, and

$$f_i \circ \gamma_{ij} = \gamma'_{ij} \circ f_j.$$

We thus conclude that the cocycles associated to a deformation form a cohomology class that depends only on the local class of the deformation.

Conversely, suppose we have a locally finite cover $\{U_i\}$ of V_0 and a cocycle $\gamma \in Z^1(V_0, \{U_i\}; \Gamma)$. Then γ_{ij} can be represented by an isomorphism from an open W_{ij} of $B \times V_0$ to another open W_{ji} , with the two open subsets both containing $\{b_0\} \times U_{ij}$. Pick a refinement $\{U'_i\}$ of the cover $\{U_i\}$, and take some neighbourhood B'' of b_0 in B small enough such that $B'' \times U'_{ij} \subset W_{ij}$ for all (i, j) , and such that the equality $\gamma_{ij} \circ \gamma_{jk} = \gamma_{ik}$ holds wherever it is defined in $B'' \times U'_{ijk}$. We thus obtain a deformation V of V_0 on B'' by gluing the $B'' \times U'_i$ via the γ_{ij} .

Finally, we can show that all the above does indeed define a bijection between the set of local classes of deformations of V_0 over (B, b_0) and $H^1(V_0; \Gamma)$. \square

2. Higher obstructions

For every open subset $U \subset V_0$, the group $\Gamma(U)$ is naturally filtered: denote by $\mathcal{F}_k(U)$ the group of vertical automorphisms that are tangent to the identity up to order $k - 1$. Then Γ becomes a filtered sheaf:

$$\Gamma = \mathcal{F}_1 \supset \mathcal{F}_2 \supset \dots \quad \text{and} \quad \bigcap \mathcal{F}_k = \{0\}.$$

Set

$$\begin{aligned} \mathcal{Q}_k &= \Gamma / \mathcal{F}_{k+1} \\ \mathcal{G}_k &= \mathcal{F}_k / \mathcal{F}_{k+1} = \text{Ker}(\mathcal{Q}_k \rightarrow \mathcal{Q}_{k-1}). \end{aligned}$$

For all k , \mathcal{G}_k is a sheaf of abelian groups, which we will write additively. If $B = \mathbb{C}$ and $b_0 = 0$ (we then speak of *the deformation in one parameter*), for all k , \mathcal{G}_k can be identified with the sheaf Θ of germs of vector fields tangent to V_0 . In the general case,

$$\mathcal{G}_k = \mathfrak{m}^k / \mathfrak{m}^{k+1} \otimes \Theta$$

where \mathfrak{m} is the maximal ideal of the point b_0 in B .

Now, if $a \in \mathcal{F}_p$ and $b \in \mathcal{F}_q$, then the commutator $aba^{-1}b^{-1}$ is in \mathcal{F}_{p+q} , and this defines a map $\mathcal{G}_p \otimes \mathcal{G}_q \rightarrow \mathcal{G}_{p+q}$ which endows $\mathcal{G} = \bigoplus \mathcal{G}_k$ with the structure of a sheaf of Lie algebras that is isomorphic to the tensor product of Θ with the graded algebra associated to the maximal ideal \mathfrak{m} of b_0 in B filtered by powers.

The exact sequence of non-abelian groups

$$0 \rightarrow \mathcal{G}_{k+1} \rightarrow \mathcal{Q}_{k+1} \rightarrow \mathcal{Q}_k \rightarrow 0$$

in which \mathcal{G}_{k+1} is a subgroup of \mathcal{Q}_{k+1} contained in its centre gives rise [3] to an exact sequence of pointed sets

$$H^1(V_0; \mathcal{Q}_{k+1}) \rightarrow H^1(V_0; \mathcal{Q}_k) \xrightarrow{\delta_k} H^2(V_0; \mathcal{G}_{k+1})$$

i.e. for an element $q \in H^1(V_0; \mathcal{Q}_k)$ to be in the image of $H^1(V_0; \mathcal{Q}_{k+1})$, it is necessary and sufficient for $\delta_k q = 0$ in $H^2(V_0; \mathcal{G}_{k+1})$. A *necessary* condition for q to be in the image of $H^1(V_0; \Gamma) \rightarrow H^1(V_0; \mathcal{Q}_k)$ is thus $\delta_k q = 0$ in $H^2(V_0; \mathcal{G}_{k+1})$.

Definition. Let $q \in H^1(V_0; \mathcal{Q}_i)$, and let $k \geq i$. We define an *obstruction of order k of the element q* to be the direct image in $H^2(V_0; \mathcal{G}_{k+1})$ under δ_k of the inverse image of q in $H^1(V_0; \mathcal{Q}_k)$. It is thus a subset of $H^2(V_0; \mathcal{G}_{k+1})$. The obstruction is said to be *trivial* if the identity element belongs to this subset. Being trivial is a necessary and sufficient condition for q to be in the image of $H^1(V_0; \mathcal{Q}_{k+1})$, and a necessary condition for q to be in the image of $H^1(V_0; \Gamma)$.

Warning. If q is not in the image of $H^1(V_0; \mathcal{Q}_k)$, then its obstruction of order k is empty, and thus non-trivial.

This definition is used most of all in the case of deformations in one parameter ($B = \mathbb{C}$ and $b_0 = 0$), where $\mathcal{G}_{k+1} = \Theta$ for all k , and $\mathcal{Q}_1 = \mathcal{G}_1 = \Theta$. The successive obstructions of an element $a \in H^1(V_0; \Theta)$ are thus subsets of $H^2(V_0; \Theta)$, and for a to be a deformation vector, it must be the case that all of its obstructions are trivial. Indeed, the element of $H^1(V_0; \Theta)$ that corresponds, under the identifications we have made ($\Theta = \mathcal{Q}_1 = \Gamma/\mathcal{F}_2$, and [Proposition 1](#)), to a deformation germ is exactly the image under the Spencer–Kodaira map ρ of the canonical basis vector of the tangent space to \mathbb{C} at 0.

II. Calculation of obstructions

1. Relation to the sheaf Ω

From now on, we work in the case of deformations in one parameter, i.e. $B = \mathbb{C}$ and $b_0 = 0$.

Let Ω be the sheaf of universal enveloping algebras of the Lie algebras of the sheaf Θ (i.e. $\Omega(U)$ is the universal enveloping algebra of $\Theta(U)$).

Then Ω contains Θ as a subsheaf, and even as a direct factor (by the Poincaré–Birkhoff–Witt Theorem in characteristic 0). For all k , consider the sheaf of algebras $\Omega_k = \Omega[t]/(t^{k+1})$. For $i \leq k$, we have a map of sheaves of sets

$$\exp_i : \Theta \rightarrow \Omega_k$$

defined by

$$\exp_i(\Theta) = \sum_p \frac{1}{M} \Theta^p t^p$$

Proposition 2. (Campbell–Hausdorff). *We can identify \mathcal{Q}_k with the sheaf of multiplicative subgroups of Ω_k generated by the images of the \exp_i for $i \leq k$.*

The proof of this proposition will not be given here. We denote by Ω_k^\times the sheaf of multiplicative subgroups of Ω_k consisting of the elements whose constant terms is 1. The

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commutative diagram of sheaves of (non-abelian) groups

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Theta & \longrightarrow & \mathcal{Q}_{k+1} & \longrightarrow & \mathcal{Q}_k \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \Omega & \longrightarrow & \Omega_{k+1}^\times & \longrightarrow & \Omega_k^\times \longrightarrow 0
 \end{array}$$

gives rise to a commutative diagram of sets

$$\begin{array}{ccc}
 \mathrm{H}^1(V_0; \mathcal{Q}_k) & \xrightarrow{\delta_k} & \mathrm{H}^2(V_0; \Theta) \\
 \downarrow & & \downarrow \\
 \mathrm{H}^1(V_0; \Omega_k^\times) & \xrightarrow{\delta_k} & \mathrm{H}^2(V_0; \Omega)
 \end{array}$$

in which $\mathrm{H}^2(V_0; \Theta)$ is a vector subspace of $\mathrm{H}^2(V_0; \Omega)$.

2. Calculation of the primary obstruction

Now let $a \in \mathrm{H}^1(V_0; \Theta)$, and let $\alpha = (\alpha_{ij})$ be a cocycle of the class a (the choice of the cocycle α does not matter, since every cocycle that is cohomologous to a deformation cocycle is itself a deformation cocycle). The corresponding multiplicative cocycle in Ω_1^\times is $(1 + \alpha_{ij}t)$. This cocycle can be lifted to Ω_i^\times as the cochain $(1 + \alpha_{ij}t)$, and we have

$$\begin{aligned}
 (1 + \alpha_{ij}t)(1 + \alpha_{jk}t) &= 1 + (\alpha_{ij} + \alpha_{jk})t + \alpha_{ij}\alpha_{jk}t^2 \\
 &= (1 + \alpha_{ik}t + \alpha_{ij}\alpha_{jk}t^2) \\
 &= (1 + \alpha_{ik}t)(1 + \alpha_{ij}\alpha_{jk}t^2).
 \end{aligned}$$

Finally, let

$$\delta_1 a = a \smile a$$

where the cup product is taken in the sheaf of algebras Ω .

Note that, if we denote by \smile the cup product taken in the sheaf of algebras opposite to Ω , i.e. defined on the level of cochains by $(\alpha \smile \beta)_{ijk} = \beta_{jk}\alpha_{ij}$, we always have that $a \smile b = -b \smile a$ in cohomology.

Consequently,

$$[a \smile a] = (a \smile a) - (a \smile a) = 2a \smile a$$

and $\delta_1 a = a \smile a = \frac{1}{2}[a \smile a]$. We thus recover, up to a factor of $\frac{1}{2}$, the obstruction defined earlier in this talk.

3. Calculation of the secondary obstruction

Now suppose that $a \smile a = 0$, so that we can find a cochain $\beta = (\beta_{ij})$ such that $\delta\beta + a \smile a = 0$, i.e.

$$\beta_{ik} = \beta_{ij} + \beta_{jk} + \alpha_{ij}\alpha_{jk}.$$

Then $(1 + \alpha_{ij}t + \beta_{ij}t^2)$ is a cocycle in Ω_2^\times , and we can choose the cochain β to be a cocycle in \mathcal{Q}_2 . | p. 4-16

This cocycle can be lifted to Ω_3^\times as the cochain $(1 + \alpha_{ij}t + \beta_{ij}t^2)$, and we have that

$$\begin{aligned} & (1 + \alpha_{ij}t + \beta_{ij}t^2)(1 + \alpha_{jk}t + \beta_{jk}t^2) \\ &= 1 + (\alpha_{ij} + \alpha_{jk})t + (\beta_{ij} + \beta_{jk} + \alpha_{ij}\alpha_{jk})t^2 + (\alpha_{ij}\beta_{jk} + \beta_{ij}\alpha_{jk})t^3 \\ &= (1 + \alpha_{ik}t + \beta_{ik}t^2)(1 + (\alpha_{ij}\beta_{jk} + \beta_{ij}\alpha_{jk})t^3). \end{aligned}$$

The secondary obstruction of a is thus the cohomology class of the cocycle $(\alpha_{ij}\beta_{jk} + \beta_{ij}\alpha_{jk}) \in Z^2(V_0; \Omega)$. This class depends on the choice of the cochain β : if we choose some other $\beta' = \beta + \theta$, where $\theta \in Z^1(V_0; \Theta)$, then the cocycle is modified by $\alpha \smile \theta + \theta \smile \alpha$, and its class by an element of $[a \smile H^1(V_0; \Theta)]$. We recover the *Massey triple product* (a, a, a) taken in the algebra Ω , but with a slightly more restrictive indetermination.

We can try to calculate this secondary obstruction without leaving the sheaf Θ , but the calculations are then much more complicated: we must take a cochain $\beta = (\beta_{ij})$ such that $\delta\beta + \frac{1}{2}[a \smile a] = 0$. Then the secondary obstruction of α is the class of the cocycle

$$[\alpha_{ij}, \beta_{jk}] + \frac{1}{6}[[\alpha_{ij}, \alpha_{jk}], \alpha_{ij} + 2\alpha_{jk}].$$

The calculation done in the sheaf of enveloping algebras Ω can be generalised to obstructions of order r : we are led to determining, by induction, cochains ω_r such that

$$\begin{cases} \omega_1 = \alpha \\ \delta\omega_r + \sum_{p+q=r} \omega_p \smile \omega_q = 0 \\ 1 + \sum_{1 \leq p \leq r} \omega_p t^p \in C^1(V_0; \mathcal{Q}_r) \end{cases}$$

4. Using spectral sequences

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Proposition 3. *Let $\varphi: V_0 \rightarrow X$ be an arbitrary map, which gives rise to a spectral sequence of graded Lie algebras*

$$H^*(X; \mathbb{R}^* \varphi \Theta) \Rightarrow H^*(V_0; \Theta).$$

Let

$$a \in H^1(X; \varphi_* \Theta) \subset H^1(V_0; \Theta).$$

If the element

$$-\frac{1}{2}[a \smile a] \in H^2(X; \varphi_* \Theta) = E_2^{2,0}$$

is non-zero, but is the image under the differential d_2 of the spectral sequence of an element $b \in E_2^{0,1}$, then the image of the secondary obstruction of a in $E_\infty^{1,1}$ consists of the elements of the form $[a, b]$. In particular, if, for all b such that $d_2 b = -\frac{1}{2}[a, a]$, we have that $[a, b] \neq 0$, then the secondary obstruction is non-trivial.

Warning. However, if $[a, b] = 0$ in $E^{1,1}$, then we can only say that the secondary obstruction comes from $E_\infty^{2,0}$, and if this group is non-zero, then we cannot conclude anything.

Proof. Let α be a cocycle on V_0 representing the class α . The element $b \in E_2^{0,1}$ can be represented by a cochain

$$\beta = (\beta_{ij}) \in C^1(V_0; \Theta)$$

such that

$$\delta\beta + \frac{1}{2}[\alpha \smile \alpha] = 0.$$

| p. 4-16

We thus obtain a cochain

$$\beta' \in C^1(V_0; \Omega)$$

such that

$$1 + \alpha t + \beta' t^2 \in C^1(V_0; \mathcal{Q}_2)$$

by setting $\beta'_{ij} = \beta_{ij} + \frac{1}{2}\alpha_{ij}^2$; this cochain satisfies $\delta\beta' + \alpha \smile \alpha = 0$. But this new cochain represents, in the $E_2^{0,1}$ term of the spectral sequence of the sheaf Ω , the same element b as the cochain β , since it differs from it by a cochain that comes from X . The secondary obstruction is thus the class of the cocycle $\alpha \smile \beta' + \beta' \smile \alpha$, which represents in the $E^{1,1}$ term of the spectral sequence the element $[a, b]$. \square

This proposition allows us to construct non-trivial examples of secondary obstructions. Consider the group N of matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

where $x, y, z \in \mathbb{C}$, and let $Y = N/\Gamma$, where Γ is the subgroup of N consisting of elements where $x, y, z \in \mathbb{Z} + i\mathbb{Z}$. Then Y is fibred over a complex torus of dimension two $T^2 \cong \mathbb{C}^2/\mathbb{Z}^4$. We find non-trivial secondary obstruction elements in $H^1(V_0; \Theta)$, where V_0 is the product of Y with a projective line D . (We use the spectral sequence obtained by projecting onto $T^2 \times D$). This variety has a “versal” deformation whose Zariski tangent space of the base B can be identified via the Spencer–Kodaira map ρ with $H^1(V_0; \Theta)$. Further, B has, at its base point b_0 , a conic singularity of degree 3, whose equation is given by the secondary obstruction.

| p. 4-19

I do not know of any examples of non-trivial secondary obstructions on varieties V_0 that satisfy $H^0(V_0; \Theta) = 0$, but some very likely exist.

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