Mixed manifolds and mixed spaces

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Translator's note.

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What follows is a translation of the French seminar talk:

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Contents

I Category of models

Let B be a topological space. We define the category $\mathcal{S}_B^n$ in the following manner: the objects of $\mathcal{S}_B^n$ are the open subsets of $B \times \mathbb{C}^n$, and a morphism $f : U \to U'$ from an open subset $U \subset B \times \mathbb{C}^n$ to an open subset $U' \subset B \times \mathbb{C}^n$ is a continuous map $f : U \to U'$ satisfying the following two conditions: *p. 2-01*

[*] https://thosgood.com/translations

1. the diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & U' \\ & {\scriptstyle \pi_1}\searrow \quad \swarrow{\scriptstyle \pi_1} & \\ & B & \end{array}$$

   commutes, where π_1 denotes the projection of $B \times \mathbb{C}^n$ to B ; and

2. for all $x \in B$, the map $f_x : U_x \to U'_x$ is holomorphic, where

$$U_x = \{z \in \mathbb{C}^n \mid (x, z) \in U\}$$

   (and similarly for U').

If B is endowed with the structure of a $\mathcal{C}^\infty$ manifold (resp. an $\mathbb{R}$-analytic manifold, resp. $\mathbb{C}$-analytic manifold), then we obtain a category $\mathcal{C}^\infty\mathcal{S}_B$ (resp. $\mathbb{R}\mathcal{S}_B$, resp. $\mathbb{C}\mathcal{S}_B$) by requiring the morphisms to be $\mathcal{C}^\infty$ (resp. $\mathbb{R}$-analytic, resp. $\mathbb{C}$-analytic).

More generally, if $f_1 : B \to B'$ is a continuous map from one topological space to another, then a *morphism of* $\mathcal{S}_{f_1}$ is a continuous map f from an object U of $\mathcal{S}_B$ to an object U' of $\mathcal{S}_{B'}$ such that

1. the diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & U' \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_1} \\ B & \xrightarrow[f_1]{} & B' \end{array}$$

   commutes ; and

2. $f_x : U_x \to U'_{f_1(x)}$ is holomorphic for all $x \in B$.

If f_1 is a $\mathcal{C}^\infty$ map from one $\mathcal{C}^\infty$ manifold to another, then f will be a morphism of $\mathcal{C}^\infty\mathcal{S}_{f_1}$ *p. 2-02*
if, further, it is a $\mathcal{C}^\infty$ map (resp. ...). We thus obtain, for every category of topological spaces, a fibred category $\mathcal{S}^n$ (resp. $\mathcal{C}^\infty\mathcal{S}^n$, resp. ...).

II The definition of mixed spaces and mixed varieties

1 First definition

Let B and V be separated spaces, and let $\pi : V \to B$ be a continuous map. The structure of a *mixed space* over B is defined on V by a system of charts $\varphi_i : U_i \to V$, where the (U_i) are objects of $\mathcal{S}^n_B$; for each i, φ_i is a homeomorphism from U_i to an open subset of V such that the diagram

$$\begin{array}{ccc} U_i & \xrightarrow{\varphi_i} & V \\ & {\scriptstyle \pi_1}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & B & \end{array}$$

commutes; finally, for all i and all j, the "change of chart" $\varphi_j^{-1} \circ \varphi_i$ is an isomorphism of $\mathcal{S}_B$ from an open subset of U_i to an open subset of U_j.

The structure thus defined is that of a $(\mathcal{C}^0,\mathbb{C})$-*mixed space*. If B is a $\mathbb{C}$-analytic space, and if the change of chart maps are all $\mathbb{C}$-analytic, then we have a $\mathbb{C}$-*analytic mixed space*. In this case, V itself is a $\mathbb{C}$-analytic space, and the fibres $V_x = \pi^{-1}(x)$ are $\mathbb{C}$-analytic submanifolds.

If B is a $\mathcal{C}^\infty$ manifold (resp. $\mathbb{R}$-analytic, resp. $\mathbb{C}$-analytic), and if the change of chart maps are all $\mathcal{C}^\infty$ (resp. ...), then we have a $(\mathcal{C}^\infty,\mathbb{C})$-*mixed manifold* (resp. $(\mathbb{R},\mathbb{C})$, resp. $(\mathbb{C},\mathbb{C})$). In this case, V itself is a manifold. Note that the notion of a $(\mathbb{C},\mathbb{C})$-mixed manifold, or a $\mathbb{C}$-analytic mixed manifold, reduces to simply having a $\mathbb{C}$-analytic manifold V endowed with a projection $\pi\colon V \to B$ onto another $\mathbb{C}$-analytic manifold such that π is of maximal rank at every point.[1]

Let $\pi\colon V \to B$ and $\pi'\colon V' \to B'$ be mixed spaces, and let $f_1\colon B \to B'$ be a continuous (resp. ...) map. Then a *morphism from V to V' over f_1* is a continuous map $f\colon V \to V'$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & V' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ B & \xrightarrow[f_1]{} & B' \end{array}$$

commutes, and such that, for any charts $\varphi_i\colon U_i \to V$ and $\varphi'_j\colon U'_j \to V'$, the map ${\varphi'_j}^{-1} \circ f \circ \varphi_i$ *p. 2-03*
is a morphism of $\mathcal{S}_{f_1}$ (resp. ...) from an open subset of U_i to U_j.

2 An equivalent definition

We now give another way of defining mixed spaces, equivalent to the above.

Given separated spaces B and V, along with a continuous map $\pi\colon V \to B$, the structure of a *pre-mixed space* consists of the structure of a $\mathbb{C}$-analytic manifold on each fibre $V_x = \pi^{-1}(x)$. Given pre-mixed spaces $\pi\colon V \to B$ and $\pi'\colon V' \to B'$, along with a continuous map $f_1\colon B \to B'$, a *morphism of pre-mixed spaces over f_1* is a continuous map $f\colon V \to V'$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & V' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ B & \xrightarrow[f_1]{} & B' \end{array}$$

commutes and induces a $\mathbb{C}$-analytic map on each fibre.

A *mixed space* is a pre-mixed space $\pi\colon V \to B$ such that every point $y \in V$ admits a neighbourhood W in V that is isomorphic as a pre-mixed space to an open subset of $B \times \mathbb{C}^n$, via an isomorphism over the identity. The morphisms of mixed spaces are the same: mixed spaces form a *full subcategory*.

3 Deformations

A mixed space $\pi\colon V \to B$ is said to be *proper* if B is locally compact and the map π is proper (i.e. the inverse image of any compact subset is compact). If it is a mixed manifold, then we can show that it is a fibred manifold that is locally trivial with respect to the

[1] *[Trans.] The more common modern nomenclature is to simply call such an object a* family of complex manifolds.

underlying $\mathcal{C}^\infty$ structure, but the previous talk shows that, in general, any two fibres are not isomorphic as $\mathbb{C}$-analytic manifolds.

Definition. Let V_0 be a compact $\mathbb{C}$-analytic manifold, B a locally compact space, and $b_0 \in B$. Then a *$\mathbb{C}$-analytic deformation of V_0 over (B, b_0)* consists of a proper $\mathbb{C}$-analytic mixed space $\pi\colon V \to B$ along with an isomorphism of $\mathbb{C}$-analytic manifolds $i\colon V_0 \to \pi^{-1}(b_0)$. *p. 2-04*

The goal of this seminar is the study, at least local, and an attempt at a classification of, $\mathbb{C}$-analytic deformations of a given compact $\mathbb{C}$-analytic manifold V_0.

Definition. Let V_0 be a compact $\mathbb{C}$-analytic manifold. A *$\mathbb{C}$-analytic deformation* $(\pi\colon V \to B, i\colon V_0 \to V)$ *of* V_0 is said to be *locally complete* if, for any other deformation $(\pi'\colon V' \to B', i'\colon V_0 \to V')$ of V_0, there exists a neighbourhood B'_1 of b'_0 in B', an analytic map $f_1\colon B'_1 \to B$ with $f_1(b'_0) \to b_0$, and a morphism of $\mathbb{C}$-analytic mixed spaces $f\colon \pi'^{-1}(B'_1) \to V$ over f_1 such that $f \circ i' = i$. The deformation is said to be *locally universal* is furthermore the germ of f_1 at b'_0 is determined uniquely by this condition.

It seems that every compact $\mathbb{C}$-analytic manifold V_0 admits a locally complete $\mathbb{C}$-analytic deformation, and a locally universal one if the group of automorphisms of V_0 is discrete.

III Vector fields

1 Study on models

Let B be a space, U an object of $\mathcal{S}_B$ (i.e. an open subset of $B \times \mathbb{C}^n$), b_0 a point of B, and set $U_0 = \pi^{-1}(b_0)$.

A holomorphic field of tangent vectors on U_0 (i.e. a holomorphic map from U_0 to $\mathbb{C}^n$) is said to be a *vertical holomorphic field* on U_0. A *vertical holomorphic field on U* is a continuous (resp. ...) map $\theta\colon U \to \mathbb{C}^n$ that induces a vertical holomorphic field on each fibre U_x. If $f\colon U \to U'$ is an isomorphism in $\mathcal{S}_B$, then the *transport $f_*\theta$ of θ by f* is defined by

$$f_*\theta(f(x,z)) = \mathrm{D}_2 f_{x,z} \cdot \theta(x,z)$$

where $\mathrm{D}_2 f_{x,z}$ is the linear map from $\mathbb{C}^n$ to itself that is tangent to f_x at the point $z \in U_x$. This is again a vertical holomorphic field, since it follows from a Cauchy integral that the matrix $\mathrm{D}f_{x,z}$ depends continuously on the pair (x,z).

Now suppose that B is a $\mathcal{C}^\infty$ manifold, just for simplicity, and let T_0 be the tangent space to B at b_0. A field of tangent vectors to U defined on U_0, i.e. a map $\omega\colon U_0 \to T_0 \times \mathbb{C}^n$, *p. 2-05* is said to be a *projectable holomorphic field* if $\omega(b_0,z) = (t_0, \theta(z))$ (where $t_0 \in T_0$ is a vector that does not depend on z, called the *projection* of the field ω) and $\theta(z)$ is a holomorphic vector field. If B is a $\mathbb{C}$-analytic space, possibly with a singularity at b_0, then we give the same definition, but with T_0 then being the *Zariski* tangent space to B at b_0, i.e. the dual of $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m}$ is the ideal of germs at b_0 of holomorphic functions on B that vanish at b_0.

If $f\colon U \to U'$ is an isomorphism of $\mathcal{C}^\infty\mathcal{S}_B$ (resp. ...), then then transport $f_*\omega$ is defined by

$$f_*\omega(f(b_0,z)) = \mathrm{D}f_{b_0,z}\omega(b_0,z)$$

where $\mathrm{D}f_{b_0,z}\colon T_0\times\mathbb{C}^n \to T_0\times\mathbb{C}^n$ is now the linear map that is tangent to f at the point (b_0,z). This is a projectable holomorphic field. Indeed, the matrix $\mathrm{D}f_{b_0,z}$ can be written as

$$\begin{pmatrix} I & 0 \\ \mathrm{D}_1 f & \mathrm{D}_2 f \end{pmatrix}$$

and

$$\mathrm{D}_1 f : T \to \mathbb{C}^n$$
$$\mathrm{D}_2 f : \mathbb{C}^n \to \mathbb{C}^n$$

both depend holomorphically on z (for $\mathrm{D}_1 f$, this follows from the fact that f_x is holomorphic for every x). By setting $f_*\omega(b_0,z') = (t_0,\theta'(z'))$, we have

$$\theta'(z') = \mathrm{D}_1 f_{b_0,z}(t_0) + \mathrm{D}_2 f_{b_0,z}(\omega(z))$$
$$\text{if } z' = f_{b_0}(z)$$

which shows that $f_*\omega$ is indeed a projectable holomorphic field.

A *projectable holomorphic field on U* is a $\mathcal{C}^\infty$ field of vectors tangent to U that induces a projectable holomorphic field on each fibre.

2 Vector fields on a mixed manifold

Let $\pi\colon V \to B$ be a $(\mathcal{C}^\infty,\mathbb{C})$-mixed manifold (resp. …, resp. a $\mathbb{C}$-analytic mixed space). By transporting along the charts, we define the notions of

- vertical holomorphic fields on an open subset of a fibre ;
- vertical holomorphic fields on a open subset of V ;
- projectable holomorphic fields on an open subset of a fibre ; and
- projectable holomorphic fields on an open subset of V.

Let ξ be a $\mathcal{C}^\infty$ vector field (resp. …) on V. By integrating ξ, we obtain a $\mathcal{C}^\infty$ map, denoted by e^ξ, from an open subset $W \subset \mathbb{R}\times V$ containing $\{0\}\times V$ (resp. $\mathbb{C}$-analytic map from an open subset $W \subset \mathbb{C}\times V$) to V, characterised by p. 2-06

(1) $e^\xi(t_1+t_2,y) = e^\xi(t_1,e^\xi(t_2,y))$, with the left-hand side being defined whenever the right-hand side is ; and

(2) $\frac{\partial}{\partial t}e^\xi(t,y)|_{0,y} = \xi(y)$.

Note that W is a mixed manifold over $\mathbb{R}\times B$ (resp. a mixed space over $\mathbb{C}\times B$).

Proposition. *For $e^\xi\colon W \to V$ to be a morphism of mixed spaces over the projection $\mathbb{R}\times B \to B$, it is necessary and sufficient for ξ to be a vertical holomorphic field. For $e^\xi\colon W \to V$ to be a morphism of mixed spaces over a map from an open subset of $\mathbb{R}\times B$ containing $\{0\}\times B$ to B, it is necessary and sufficient for ξ to be a projectable holomorphic field.*

The proof is left to the reader.

IV The Spencer–Kodaira map

Let $\pi\colon V \to B$ be a mixed manifold (resp. a $\mathbb{C}$-analytic mixed space), $b \in B$, and $V_0 = \pi^{-1}(b_0)$. Let T_0 be the tangent space to B at b_0 (resp. the Zariski tangent space). We introduce the following sheaves on V_0:

Θ_0: the sheaf of germs of vertical holomorphic fields on V_0 ;

Π_0: the sheaf of germs of locally projectable holomorphic fields on V_0 ; and

Λ_0: the sheaf $\pi^* T_0$, i.e. the sheaf of germs of locally constant maps from V_0 to T_0.

We have an exact sequence of sheaves on V_0

$$0 \to \Theta_0 \to \Pi_0 \to \Lambda_0 \to 0$$

that gives rise to the long exact sequence in cohomology

$$\ldots \to \mathrm{H}^0(V_0;\Pi_0) \to \mathrm{H}^0(V_0;\Lambda_0) \xrightarrow{\delta} \mathrm{H}^1(V_0;\Theta_0) \to \ldots.$$

We also have a canonical map *p. 2-07*

$$\iota\colon T_0 \to \mathrm{H}^0(V_0;\Lambda_0)$$

that is injective if V_0 is non-empty, and surjective if V_0 is connected.

Definition. The *Spencer–Kodaira map* is the composition

$$\rho_0 = \delta \circ \iota\colon T_0 \to \mathrm{H}^1(V_0;\Theta_0).$$

This map is an essential tool in the local study of deformations of $\mathbb{C}$-analytic varieties. Note that Θ_0 is exactly the sheaf of germs of holomorphic fields of tangent vectors to V_0, and thus depends only on V_0, while T_0 depends only on the base. Also, Θ_0 is a coherent analytic sheaf on V_0, and, if V_0 is compact, then $\mathrm{H}^1(V_0;\Theta_0)$ is a finite-dimensional vector space over $\mathbb{C}$ [1]. We thus see that, in this case (which is the only case where we can say anything non-trivial), ρ_0 might be possible to calculate.

It is clear that, if the given mixed manifold is trivial (i.e. if $V = B \times V_0$, with π being the projection to B), then the map ρ_0 is zero. The next talk aims to show that, in a certain sense, ρ indicates the non-triviality of V in a neighbourhood of V_0.

References

[1] Cartan, H. Un théorème de finitude. *Séminaire H. Cartan* **6** (1953–54), Talk no. 17.

[2] Kodaira, K. and Spencer, D. On deformation of complex analytic structures, I. *Annals of Math.* **67** (1958), 328–401.