Regular deformations

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Translator's note.

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What follows is a translation of the French seminar talk:

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All throughout this talk, B is a $\mathcal{C}^\infty$ manifold (resp. $\mathbb{R}$-analytic, resp. $\mathbb{C}$-analytic); $\pi: V \to B$ denotes a proper mixed manifold; b_0 is a point of B; and $V_0 = \pi^{-1}(b_0)$ is thus a compact $\mathbb{C}$-analytic manifold.

I The map $\widetilde{\rho}$

Let $\widetilde{\Theta}$ (resp. $\widetilde{\Pi}$) be the sheaf of germs of vertical holomorphic (resp. locally projectable holomorphic) vector fields on V. The quotient sheaf $\widetilde{\Lambda} = \widetilde{\Pi}/\widetilde{\Theta}$ is exactly the inverse image under π of the sheaf $\widetilde{T}$ of germs of $\mathcal{C}^\infty$ fields (resp. ...) of tangent vectors on B.

* https://thosgood.com/translations

For every open subset U of B, set $V_U = \pi^{-1}(U)$. The exact sequence

$$0 \to \widetilde{\Theta} \to \widetilde{\Pi} \to \widetilde{\Lambda} \to 0$$

of sheaves on V_U gives rise to a homomorphism

$$\widetilde{\rho}_U \colon \mathrm{H}^0(U;\widetilde{T}) \xrightarrow{\pi_*} \mathrm{H}^0(V_U;\widetilde{\Lambda}) \xrightarrow{\delta} \mathrm{H}^1(V_U;\widetilde{\Theta}).$$

Let $\mathrm{R}^1\pi_*\widetilde{\Theta}$ be the sheaf on B defined by the presheaf $U \mapsto \mathrm{H}^1(V_U;\widetilde{\Theta})$. Then $\widetilde{\rho}$ becomes a homomorphism of sheaves on B:

$$\widetilde{\rho} \colon \widetilde{T} \to \mathrm{R}^1\pi_*\widetilde{\Theta}.$$

In particular, we have a homomorphism

$$\widetilde{\rho}_0 \colon \widetilde{T}_0 \to \mathrm{R}^1\pi_*\widetilde{\Theta} = \mathrm{H}^1(V_0;\widetilde{\Theta})$$

where $\widetilde{T}_0$ is the vector space of germs at b_0 of fields of tangent vectors to B. Finally, we have a commutative diagram *p. 3-02*

$$\begin{array}{ccc} \widetilde{T}_0 & \xrightarrow{\widetilde{\rho}_0} & \mathrm{H}^1(V_0;\widetilde{\Theta}) \\ {\scriptstyle\varepsilon}\downarrow & & \downarrow{\scriptstyle\varepsilon} \\ T_0 & \xrightarrow[\rho_0]{} & \mathrm{H}^1(V_0;\Theta_0) \end{array}$$

where ρ_0 is the Spencer–Kodaira map [2].

Theorem 1. *For the proper mixed manifold $\pi \colon V \to B$ to be locally trivial in a neighbourhood of the point $b_0 \in B$, it is necessary and sufficient for the map $\widetilde{\rho}_0 \colon \widetilde{T}_0 \to \mathrm{H}^1(V_0;\widetilde{\Theta})$ to be zero.*

Proof.

(Necessary). If $\pi \colon V \to B$ is locally trivial at b_0, then, for every open subset U of B over which V is trivial, we have $\widetilde{\Pi} = \widetilde{\Lambda} \oplus \widetilde{\Theta}$ on V_U, and so $\delta \colon \mathrm{H}^0(V_U;\widetilde{\Lambda}) \to \mathrm{H}^0(V_U;\widetilde{\Theta})$ is zero.

(Sufficient). Let $(\eta_1,\ldots,\eta_p)$ be $\mathcal{C}^\infty$ vector fields (resp. ...) on a neighbourhood of b_0 in B, such that $(\eta_1(b_0),\ldots,\eta_p(b_0))$ forms a basis of the tangent space T_0 to B at b_0. It then follows from the hypothesis that the map

$$\mathrm{H}^0(V_0;\widetilde{\Pi}) \to \mathrm{H}^0(V_0;\widetilde{\Lambda})$$

is surjective.

So let $(\xi_1,\ldots,\xi_p)$ be projectable holomorphic vector fields on a neighbourhood of V_0 in V, that project to $(\eta_1,\ldots,\eta_p)$. Let f be the map defined on a neighbourhood of $\{0\} \times V_0$ in $\mathbb{R}^p \times V_0$ (resp. $\mathbb{C}^p \times V_0$) by

$$f(t_1,\ldots,t_p,y) = e^{\xi_1}(t_1, e^{\xi_2}(\ldots, e^{\xi_p}(t_p,y)\ldots)).$$

It follows from the proposition stated in [1, §III.2] that f induces an isomorphism of mixed manifolds from $U \times V_0$ to $\pi^{-1}(f_1(U))$ over f_1, where U is a sufficiently small cubical neighbourhood of 0 in $\mathbb{R}^p$, and f_1 is the map from U to B defined by

$$f_1(t_1,\ldots,t_p) = e^{\eta_1}(t_1, e^{\eta_2}(\ldots, e^{\eta_p}(t_p,b_0)\ldots)),$$

which proves the theorem. *p. 3-03*

□

II The regular case

For all $b \in B$, set $V_b = \pi^{-1}(b)$. Consider the family $\{\mathrm{H}^1(V_b;\Theta_b)\}_{b\in B}$ of finite-dimensional $\mathbb{C}$-vector spaces, and, for all $b \in B$, the map

$$\varepsilon_b \colon \mathrm{H}^1(V_b;\widetilde{\Theta}) \to \mathrm{H}^1(V_b;\Theta_b).$$

For every open subset $U \subset B$, we have a map

$$\widetilde{\varepsilon}_U \colon \mathrm{H}^1(V_U;\widetilde{\Theta}) \to \prod_{b\in U} \mathrm{H}^1(V_b;\Theta_B)$$

that defines, by varying U, a homomorphism from the sheaf $\mathrm{R}^1\pi_*\widetilde{\Theta}$ to the sheaf Φ on B defined by $\Phi(U) = \prod_{b\in U} \mathrm{H}^1(V_b;\Theta_b)$.

Definition. We say that the proper mixed manifold $\pi\colon V \to B$ is *regular* if

1. the dimension of $\mathrm{H}^1(V_b;\Theta_b)$ does not depend on the point $b \in B$; and
2. we can endow $E = \bigcup_{b\in B} \mathrm{H}^1(V_b;\Theta_b)$ with the structure of a $\mathcal{C}^\infty$ vector bundle (resp. ...) such that $\widetilde{\varepsilon}$ is an isomorphism from the sheaf $\mathrm{R}^1\pi_*\widetilde{\Theta}$ to the sheaf of germs of $\mathcal{C}^\infty$ sections (resp. ...) of the bundle E.

In fact, Kodaira and Spencer have shown [2] that, by identifying the H^1 spaces with spaces of harmonic forms, condition 2 is a consequence of condition 1.

Then Theorem 1 has the following corollary:

Proposition 1. *For the proper mixed manifold $\pi\colon V \to B$ to be locally trivial, it is necessary and sufficient for it to be regular and, for all $b \in B$, for the Spencer–Kodaira map*

$$\rho_b \colon T_b \to \mathrm{H}^1(V_b;\Theta_b)$$

to be zero.

Indeed, since $\widetilde{\varepsilon}$ is injective, this condition implies that the map *p. 3-04*

$$\widetilde{\rho}_b \colon \widetilde{T}_b \to \mathrm{H}^1(V_b;\widetilde{\Theta})$$

is zero for all b.

At the end of this talk, we will construct a counter-example which shows that it is necessary to assume that the mixed manifold is regular.

III An example of non-regular deformation: Hopf manifolds

1 Hopf manifolds

Let $n \geqslant 2$ be an integer, and let b be an $(n\times n)$ matrix with coefficients in $\mathbb{C}$, whose eigenvalues are all of modulus > 1. The free group $L(b)$ generated by b acts freely on $\widetilde{V} = \mathbb{C}^n \setminus \{0\}$, and the quotient space $\widetilde{V}/L(b)$, which we call the *Hopf manifold defined by b*, is a compact $\mathbb{C}$-analytic manifold that is homeomorphic to $S^{2n-1} \times S^1$.

Note that V_b and $V_{b'}$ are isomorphic if and only if there exists some a such that $b' = aba^{-1}$ or $b' = ab^{-1}a^{-1}$ (cf. Appendix).

Let Θ be the sheaf of germs of holomorphic fields of tangent vectors on V_b.

Proposition 2. $\mathrm{H}^0(V_b;\Theta)$ *can be identified with the vector space of matrices that commute with b, and* $\mathrm{H}^1(V_b;\Theta)$ *has the same dimension as this vector space.*

Proof. If X is a vector field on an open subset $U \subset \widetilde{V}$, then $b_*(X)$ is the vector field on the open subset $b(U)$ given by transporting via b, i.e. $b_*X(u) = bX(b^{-1}u)$. Let $\mathcal{U} = \{U_i\}$ be a cover of V by simply connected Stein open subsets; for all i, set $\widetilde{U}_i = \chi^{-1}\{U_i\}$, where χ is the canonical map from $\widetilde{V}$ to V_b. The cover $\widetilde{\mathcal{U}} = \{\widetilde{U}_i\}$ of $\widetilde{V}$ consists of Stein open subsets that are invariant under b (not necessarily connected, but this doesn't matter). Then b_* defines a map, again denoted by b_*, from the group of cochains $C^\bullet(\widetilde{V},\widetilde{U};\Theta)$ to itself.

Lemma 1. *We have the exact sequence*

$$0 \to C^\bullet(V_b,\mathcal{U};\Theta) \xrightarrow{\chi^*} C^\bullet(\widetilde{V},\widetilde{U};\Theta) \xrightarrow{1-b_*} C^\bullet(\widetilde{V},\widetilde{U};\Theta) \to 0.$$

Proof. The only thing that we need to verify is that the map $1-b_*$ is surjective. For all $(i_0,\ldots,i_q)$, let $U'_{i_0,\ldots,i_q}$ be an open subset of $\widetilde{V}$ such that p. 3-05

$$\chi\colon U'_{i_0,\ldots,i_q} \to U_{i_0,\ldots,i_q}$$

is a homeomorphism. The $\widetilde{U}_{i_0,\ldots,i_q}$ is a disjoint union of the $b_*^p U'_{i_0,\ldots,i_q}$, where $p \in \mathbb{Z}$, and every $\gamma \in C^q(\widetilde{V},\widetilde{U};\Theta)$ can be written in the form $\gamma = \gamma_1 - \gamma_2$, with $\gamma_1 = 0$ on $b^p(U'_{i_0,\ldots,i_q})$ for $p<0$, and $\gamma_2 = 0$ for $p \geqslant 0$. Set

$$\beta = \sum_{p\geqslant 0} b_*^p\gamma_1 + \sum_{p<0} b_*^p\gamma_2$$

(which is a locally finite sum). Then $\beta - b_*\beta = \gamma$, whence Lemma 1. □

Now, to finish the proof of Proposition 2. From Lemma 1, we have the following exact sequence:

$$0 \to \mathrm{H}^0(V_b;\Theta) \xrightarrow{\chi^*} \mathrm{H}^0(\widetilde{V};\Theta) \xrightarrow{1-b_*} \mathrm{H}^0(\widetilde{V};\Theta) \xrightarrow{\delta_*} \mathrm{H}^1(V_b;\Theta) \xrightarrow{\chi^*} \mathrm{H}^1(\widetilde{V}y\Theta) \xrightarrow{1-b_*} \mathrm{H}^1(\widetilde{V};\Theta).$$

We can show that

$$\chi^*\colon \mathrm{H}^1(V_b;\Theta) \to \mathrm{H}^1(\widetilde{V};\Theta)$$

is zero: if $n>2$, it is evident, since $\mathrm{H}^1(\widetilde{V};\Theta) = 0$; if $n=2$, then a direct calculation on the cochains of a cover of $\widetilde{V}$ by two Stein open subsets shows that

$$1-b_*\colon \mathrm{H}^1(\widetilde{V};\Theta) \to \mathrm{H}^1(\widetilde{V};\Theta)$$

is bijective.

Now $\mathrm{H}^0(\widetilde{V};\Theta)$ is the space of holomorphic vector fields on $\widetilde{V}$, but such a field extends to a holomorphic vector field on $\mathbb{C}^n$, and $\mathrm{H}^0(\widetilde{V},\Theta) = L \oplus M$, where L is the space of fields of linear vectors, and M is the space of fields of second-order vectors at 0. The subspaces L and M are invariant under b_*, and $1-b_*\colon M \to M$ is an isomorphism. Then Proposition 2 follows from remarking that, if an element of L is represented by a matrix a, then $b_*a = bab^{-1}$. □

2 Mixed manifolds whose fibres are Hopf manifolds

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Let B be the set of all $(n\times n)$ matrices with coefficients in $\mathbb{C}$ with eigenvalues all of modulus >1. This is an open subset of $\mathbb{C}^{n^2}$. Let α be the transformation from $B\times\widetilde{V}$ to itself defined by $\alpha(b,x)=(b,b(x))$. The free group $L(\alpha)$ generated by α acts linearly on $B\times\widetilde{V}$, and the quotient $V=B\times\widetilde{V}/L(\alpha)$ is a $\mathbb{C}$-analytic manifold. By endowing it with the projection $\pi\colon V\to B$ induced by the projection $\pi_1\colon B\times\widetilde{V}\to B$ after passing to the quotient, we obtain a $\mathbb{C}$-analytic mixed manifold that is proper, but not regular. Indeed, condition 1 of the definition of regular mixed manifolds is not satisfied: for example, for $n=2$, the dimension of $\mathrm{H}^1(V_b;\Theta)$ is 4 if b is a scalar matrix, but 2 in all other cases.

Note that the dimension of $\mathrm{H}^1(V_b;\Theta_b)$ is an upper semi-continuous function of b, and that the set of b such that $\dim\mathrm{H}^1(V_b;\Theta_b)\geqslant k$ is a closed analytic subspace of B. This is a general result, that we hope to be able to prove in a later talk of this seminar.

3 Calculation of ρ

We have $T_b=\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^n)=L\subset\mathrm{H}^0(\widetilde{V};\Theta)$, and we defined, to prove Proposition 2, a surjective map $\delta_*\colon L\to\mathrm{H}^1(V_b;\Theta)$.

Proposition 3. *The Spencer–Kodaira map ρ is given, for the mixed manifold studied in this section, by*

$$\rho(a)=\delta_*(ab^{-1}).$$

In particular, it is surjective, and its kernel is the space of matrices of the form $[\ell,b]$ for $\ell\in L$.

Proof. Let $a\in T_b=L$. Let $\{U_i\}$ be a cover of V_b by simply connected Stein open subsets, and, for each i, let U_i' be a connected component of $\widetilde{U}_i$.

Let η_i' be the projectable holomorphic field on U_i' defined by $\eta_i'(x)=(a,0)$; let $\widetilde{\eta}_i$ be the projectable holomorphic field on $\widetilde{U}_i$ defined by $\widetilde{\eta}_i=\alpha_*^k\eta_i'$ on $b^k(U_i')$; and let η_i be the projectable holomorphic field on U_i corresponding to $\widetilde{\eta}_i$. By definition, $\rho(a)$ is the cohomology class of the cochain $\{\theta_{ij}\}$, where $\theta_{ij}=\eta_j-\eta_i$ is a vertical holomorphic field on U_{ij}. p. 3-07

Set $\widetilde{\eta}_i(x)=(a,\beta_i(x))$. Then $\beta\in C^0(\widetilde{V};\Theta)$, and we have $(1-b_*)\beta=ab^{-1}\in L\subset\mathrm{H}^0(\widetilde{V};\Theta)$. Indeed, $\alpha_*\eta=\eta$, $\alpha_*\eta_i(b_{-1}x)=\eta_i(x)$, and

$$\alpha_*(a,\beta(b^{-1}x))=(a,\beta(x)),$$

whence

$$ab^{-1}x+b\cdot\beta(b^{-1}x)=\beta(x).$$

We thus deduce that $\theta=\delta_*(ab^{-1})$, which proves Proposition 3. □

4 A counter-example

Take $n=2$, and $\sigma\in\mathbb{C}$ such that $|\sigma|>1$. Let $B'\subset B$ be the set of matrices of the form

$$\begin{pmatrix}\sigma & t\\ 0 & \sigma\end{pmatrix}$$

where $t \in \mathbb{C}$, and let $V' = \pi^{-1}(B')$ be the mixed manifold induced by V over V'; now B' is a line, and its tangent space T'_b at b is generated, for all b, by $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. It follows from Proposition 3 that the Spencer–Kodaira map

$$\rho' \colon T_b(B') \to \mathrm{H}^1(V_b;\Theta)$$

is zero if and only if

$$b \neq b_0 = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix}$$

since, if $b \neq b_0$, then $a = [\ell, b]$, where $\ell = \begin{pmatrix} t^{-1} & 0 \\ 0 & 0 \end{pmatrix}$; and if $b = b_0$, then ρ' is injective.

We can also see that V' is trivial on $B' \setminus \{b_0\}$.

Let $\varphi \colon \mathbb{C} \to B' \subset B$ be the map defined by

$$\varphi(t) = \begin{pmatrix} \sigma & t^2 \\ 0 & \sigma \end{pmatrix}$$

and let V^φ be the mixed manifold given by the inverse image of V under φ. The Spencer–Kodaira map ρ_t^φ from $\mathbb{C}$ to $\mathrm{H}^1(V_{\varphi(t)};\Theta)$ is the composition p. 3-08

$$\rho'_{\varphi(t)} \circ \mathrm{D}\varphi \colon \mathbb{C} \to T'_{\varphi(t)} \to \mathrm{H}^1(V_{\varphi(t)};\Theta),$$

and this is zero for all t, since, if $t \neq 0$, then $\rho'_{\varphi(t)}$ is zero; and, if $t = 0$, then $\mathrm{D}\varphi$ is zero.

However, the mixed manifold V^φ is not locally trivial, since V_0^φ is not isomorphic to V_t^φ for $t \neq 0$.

5 Question (K. Srinivasacharyulu)

We know that the Hopf manifolds are non-Kähler, and thus non-algebraic. For $n = 2$, the manifold V_b admits non-constant meromorphic functions if and only if b can be diagonalised with eigenvalues σ_1 and σ_2 satisfying $\sigma_1^p = \sigma_2^q$ for some integers p and q (and there is then the function $x_1^p x_2^{-q}$). The set of b satisfying this property is neither open nor closed, but it is a countable union of closed analytic subspaces. An analogous phenomenon arises for deformations of complex tori. Is this result general?

Appendix

With the notation of III.1, let $f : V_b \to V_{b'}$ be an isomorphism of $\mathbb{C}$-analytic manifolds. This lifts to an isomorphism of universal coverings

$$\widetilde{f} \colon \mathbb{C}^n \setminus \{0\} \to \mathbb{C}^n \setminus \{0\}.$$

By Hartog, $\widetilde{f}$ extends to an isomorphism $g \colon \mathbb{C}^n \to \mathbb{C}^n$. We necessarily have

$$g(bz) = (b')^k g(z) \qquad (*)$$

where $z \in \mathbb{C}^n$, and k is an integer; the same property, applied to the inverse map of g, shows that $k = \pm 1$. Let a be the linear map that is tangent to g at the origin; the identity $(*)$ then gives

$$ab = (b')^k a$$
$$k = \pm 1$$

whence

$$b' = aba^{-1} \quad \text{or} \quad b' = ab^{-1}a^{-1}.$$

References

[1] Douady, A. Variétés et espaces mixtes *Séminaire H. Cartan* **13** (1960–61), Talk no. 2.

[2] Kodaira, K. and Spencer, D. On deformation of complex analytic structures, I. *Annals of Math.* **67** (1958), 328–401.