

# The primary obstruction to deformation

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**Translator’s note.**

*This text is one of a series\* of translations of various papers into English. The translator takes full responsibility for any errors introduced in the passage from one language to another, and claims no rights to any of the mathematical content herein.*

*What follows is a translation of the French seminar talk:*

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## Introduction

Let  $V_0$  be a compact complex-analytic manifold, and let  $\Theta$  be the sheaf of germs of holomorphic fields of tangent vectors. We ask the following question: given an element  $a \in H^1(V_0, \Theta)$ , does there exist a deformation of  $V_0$ , with a non-singular base (i.e. a fibred mixed manifold  $\pi: V \rightarrow B$ , with  $b_0 \in B$ , along with an isomorphism  $V_0 \xrightarrow{\cong} \pi^{-1}(b_0)$ ), such

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that  $a$  is the image, under the map  $\rho$  defined in [Talk no. 2], of a vector  $v$  that is tangent to  $B$  at  $b_0$ ? An element  $a \in H^1(V_0, \Theta)$  for which the answer is positive is called a *deformation vector*. We will give a necessary condition for  $a$  to be a deformation vector; this condition is written  $[a \smile a] = 0$ . We will then give an example where this condition is not satisfied.

## I Exact sequences of sheaves of algebras

Let  $K$  be a commutative ring, and let  $\Phi$ ,  $\Phi_1$ , and  $\Phi_2$  be sheaves of  $K$ -modules on some space  $X$ , and suppose that we have some given homomorphism  $\Phi_1 \otimes \Phi_2 \rightarrow \Phi$ , written as a product. We define, for any cover  $\mathcal{U}$  of  $X$ , the *cup product*

$$\smile: C^p(X, \mathcal{U}; \Phi_1) \otimes C^q(X, \mathcal{U}; \Phi_2) \rightarrow C^{p+q}(X, \mathcal{U}; \Phi)$$

by the formula

$$(\alpha \smile \beta)_{i_0, \dots, i_{p+q}} = \alpha_{i_0, \dots, i_p} \cdot \beta_{i_p, \dots, i_{p+q}}.$$

We have the relation

$$d(\alpha \smile \beta) = d\alpha \smile \beta + (-1)^p \alpha \smile d\beta.$$

This induces a cup product on the cohomology of the cover  $\mathcal{U}$ , and, by passing to the inductive limit over open covers, a cup product

$$\smile: H^p(X; \Phi_1) \otimes H^q(X; \Phi_2) \rightarrow H^{p+q}(X; \Phi).$$

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**Definition.** A *sheaf of algebras* on  $X$  is a sheaf of modules  $\Phi$  on  $X$  endowed with a product  $\Phi \otimes \Phi \rightarrow \Phi$  (which we do not assume to be either commutative nor associative).

If  $f: \Phi \rightarrow \Psi$  is a homomorphism of sheaves of algebras, then the kernel  $\Phi'$  of  $f$  is a sheaf of two-sided ideals of  $\Phi$ , i.e. we have products  $\Phi' \otimes \Phi \rightarrow \Phi'$  and  $\Phi \otimes \Phi' \rightarrow \Phi'$  such that the two diagrams

$$\begin{array}{ccc} \Phi' \otimes \Phi & \longrightarrow & \Phi' \\ \downarrow & & \downarrow \\ \Phi \otimes \Phi & \longrightarrow & \Phi \end{array} \quad \begin{array}{ccc} \Phi \otimes \Phi' & \longrightarrow & \Phi' \\ \downarrow & & \downarrow \\ \Phi \otimes \Phi & \longrightarrow & \Phi \end{array}$$

both commute.

**Proposition 1.** Let  $0 \rightarrow \Phi' \rightarrow \Phi \rightarrow \Phi'' \rightarrow 0$  be an exact sequence of sheaves of algebras on  $X$ ; let  $a \in H^p(X; \Phi'')$ . Then  $\delta a \in H^{p+1}(X; \Phi')$ , and, for any class  $b \in H^q(X; \Phi')$ , we have

$$\delta a \smile b = 0.$$

*Proof.* Let  $\mathcal{U}$  be a cover of  $X$  such that  $a$  and  $b$  are represented by cocycles  $\alpha$  and  $\beta$  (respectively), and such that  $\alpha$  lifts to a cochain  $\eta \in C^p(X, \mathcal{U}; \Phi)$ . Then  $\delta\eta$  is a cocycle in  $C^{p+1}(X, \mathcal{U}; \Phi')$  whose class in  $H^{p+1}(X; \Phi')$  is, by definition,  $\delta a$ , and  $\delta a \smile b$  is the class of  $\delta\eta \smile \beta$ . But  $\delta(\eta \smile \beta) = \delta\eta \smile \beta$ , and  $\eta \smile \beta$  is a cochain in  $C^{p+q}(X, \mathcal{U}; \Phi')$ , since  $\Phi'$  is a sheaf of ideals. So the cocycle  $\delta\eta \smile \beta$  is cohomologous to 0 in  $H^{p+q+1}(X; \Phi')$ , which proves the proposition.  $\square$

## II The primary obstruction

Let  $V_0$  be a complex-analytic manifold, and  $\Theta_0$  the sheaf of germs of holomorphic fields of tangent vectors. Then  $\Theta_0$  is a sheaf of Lie algebras, and, if  $a, b \in H^*(V_0, \Theta_0)$ , then we denote by  $[a \smile b]$  the cup product defined by the bracket  $[-, -]: \Theta_0 \otimes \Theta_0 \rightarrow \Theta_0$ . It satisfies

$$[b \smile a] = (-1)^{pq+1}[a \smile b]$$

for  $a \in H^p(V_0, \Theta_0)$  and  $b \in H^q(V_0, \Theta_0)$ .

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**Theorem 1.** *Let  $\pi: V \rightarrow B$  be a mixed manifold,  $b_0$  a point of  $B$ ,  $V_0 = \pi^{-1}(b_0)$ , and let  $\rho_0: T_0 \rightarrow H^1(V_0, \Theta_0)$  be Spencer-Kodaira map. Then, if  $u$  and  $v$  are tangent vectors of  $B$  at  $b_0$ , we have*

$$[\rho_0(u) \smile \rho_0(v)] = 0.$$

**Corollary.** *Let  $V_0$  be a complex-analytic manifold, and  $\Theta$  the sheaf of germs of holomorphic fields of tangent vectors of  $V_0$ . If  $a \in H^1(V_0, \Theta)$  is a deformation vector, then*

$$[a \smile a] = 0.$$

*Proof of the Corollary.* This is simply a particular case of [Theorem 1](#); note that  $[a \smile b]$  is a symmetric bilinear map from  $H^1 \otimes H^1$  to  $H^2$ , and that we are in characteristic  $0 \neq 2$ .  $\square$

*Proof of Theorem 1.* Consider the following sheaves on  $V_0$ :

$\Theta_0$ : the sheaf of germs of vertical holomorphic fields on  $V_0$  ;

$\tilde{\Theta}_0$ : the sheaf of germs of vertical holomorphic fields on  $V$  ;

$\Pi_0$ : the sheaf of germs of locally projectable holomorphic fields on  $V_0$  ;

$\tilde{\Pi}_0$ : the sheaf of germs of locally projectable holomorphic fields on  $V$  ;

$\Lambda_0$ : the sheaf  $\pi^* T_0$ , where  $T_0$  is the tangent space of  $B$  at  $b_0$  ; and

$\tilde{\Lambda}_0$ : the sheaf  $\pi^* \tilde{T}_0$ , where  $\tilde{T}_0$  is the space of germs at  $b_0$  of fields on  $B$  of tangent vectors of  $B$ .

We have the following diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{\Theta}_0 & \longrightarrow & \tilde{\Pi}_0 & \longrightarrow & \tilde{\Lambda}_0 \longrightarrow 0 \\ & & \downarrow \epsilon & & \downarrow \epsilon & & \downarrow \epsilon \\ 0 & \longrightarrow & \Theta_0 & \longrightarrow & \Pi_0 & \longrightarrow & \Lambda_0 \longrightarrow 0 \end{array}$$

whence we obtain the following commutative diagram:

| p. 4-04

$$\begin{array}{ccc} \tilde{T}_0 & \xrightarrow{\tilde{\rho}} & H^1(V_0, \tilde{\Theta}_0) \\ \epsilon \downarrow & & \downarrow \epsilon \\ T_0 & \xrightarrow{\rho} & H^1(V_0, \Theta_0) \end{array}$$

Let  $u, v \in T_0$  be fixed tangent vectors of  $B$  at  $b_0$ . We can always find vector fields  $\tilde{u}$  and  $\tilde{v}$  on  $B$  that take the values  $u$  and  $v$  (respectively) at  $b_0$ ;  $\epsilon(\tilde{u}) = u$  and  $\epsilon(\tilde{v}) = v$ . The exact sequence

$$0 \rightarrow \tilde{\Theta}_0 \rightarrow \tilde{\Pi}_0 \rightarrow \tilde{\Lambda}_0 \rightarrow 0$$

is a sequence of homomorphisms of sheaves of Lie algebras, and so

$$[\tilde{\rho}(\tilde{u}) \smile \tilde{\rho}(\tilde{v})] = 0$$

by [Proposition 1](#). But  $\epsilon: \tilde{\Theta}_0 \rightarrow \Theta_0$  is also a homomorphism of sheaves of Lie algebras, and the diagram

$$\begin{array}{ccc} \mathrm{H}^1(V_0, \tilde{\Theta}_0) \otimes \mathrm{H}^1(V_0, \tilde{\Theta}_0) & \xrightarrow{[-\smile-]} & \mathrm{H}^2(V_0, \tilde{\Theta}_0) \\ \epsilon \otimes \epsilon \downarrow & & \downarrow \epsilon \\ \mathrm{H}^1(V_0, \Theta_0) \otimes \mathrm{H}^1(V_0, \tilde{\Theta}_0) & \xrightarrow{[-\smile-]} & \mathrm{H}^2(V_0, \Theta_0) \end{array}$$

commutes. We thus deduce that  $[\rho(u) \smile \rho(v)] = 0$ . □

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**Remarks.**

1. We make essential use of the fact that  $\epsilon: \tilde{T}_0 \rightarrow T_0$  is surjective, and thus of the fact that  $B$  has no singularities.
2. We actually have  $[\rho(u) \smile b] = 0$  for all  $u \in T_0$ , for any class  $b \in \mathrm{H}^1(V_0, \Theta_0)$  that is in the image of  $\mathrm{H}^1(V_0, \tilde{\Theta}_0)$  under  $\epsilon$ . In particular, for an element  $a \in \mathrm{H}^1(V_0, \Theta_0)$  to be a regular deformation vector (in the sense of [Talk no. 3]), it is necessary and sufficient for  $[a \smile b] = 0$  for all  $b \in \mathrm{H}^1(V_0, \Theta_0)$ .

If  $V_0$  is a compact complex-analytic manifold, and  $a \in \mathrm{H}^1(V_0, \Theta)$ , then we call  $[a \smile a] \in \mathrm{H}^2(V_0, \Theta)$  the *primary obstruction* to the deformation of  $V_0$  along  $a$ . For  $a$  to be a deformation vector, it is necessary that this primary obstruction be zero; but it is not sufficient: we can define a sequence of set-theoretic maps  $\omega_n$ , called *obstructions*, with  $\omega_1: \mathrm{H}^1(V_0, \Theta) \rightarrow \mathrm{H}^2(V_0, \Theta)$  given by  $\omega_1(a) = [a \smile a]$ , and with  $\omega_{k+1}$  defined on the subset of  $\mathrm{H}^1(V_0, \Theta)$  where  $\omega_k$  vanishes, with values in varying quotients<sup>1</sup> of  $\mathrm{H}^2(V_0, \Theta)$ , and a necessary condition for  $a$  to be a deformation vector is that all the  $\omega_k(a)$  be defined and real. I do not know if *this* condition is sufficient. Kodaira, Spencer, and Nijenhuis [4] have shown that, if  $\mathrm{H}^2(V_0, \Theta) = 0$ , then every element of  $\mathrm{H}^1(V_0, \Theta)$  is a deformation vector. In this case, we even have a locally universal deformation whose base is a manifold, and  $\rho$  is an isomorphism from the tangent space of this manifold to  $\mathrm{H}^1(V_0, \Theta)$

### III An example of obstruction

#### 1 The manifold $V_0$

Let  $X = E/\Gamma$  be a 2-dimensional complex torus, i.e.  $E \cong \mathbb{C}^2$  and  $\Gamma \cong \mathbb{Z}^4$ , and let  $D$  be the projective line  $\mathbb{P}^1\mathbb{C}$ . Set  $V_0 = X \times D$ . The sheaf  $\Theta$  of holomorphic fields of tangent vectors

<sup>1</sup>See the appendix.

of  $V_0$  is the direct sum of the sheaves of Lie algebras  $\Theta_1$  and  $\Theta_2$ , where

$$\begin{aligned}\Theta_1 &= \mathcal{O} \otimes_{\mathcal{O}_X} \pi_1^* \Theta_X \\ \Theta_2 &= \mathcal{O} \otimes_{\mathcal{O}_D} \pi_2^* \Theta_D\end{aligned}$$

where  $\pi_1: V_0 \rightarrow X$  and  $\pi_2: V_0 \rightarrow D$  are the projections,  $\mathcal{O}$ ,  $\mathcal{O}_X$ , and  $\mathcal{D}$  are the structure sheaves (sheaves of local rings), and  $\Theta_X$  and  $\Theta_D$  are the sheaves of germs of holomorphic fields of tangent vectors of  $X$  and  $D$  (respectively). We are mostly interested in  $\Theta_2$ . Also,  $H^1(V_0, \Theta_2)$  is given by the Künneth exact sequence:

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$$0 \rightarrow H^0(X, \mathcal{O}_X) \otimes H^1(D, \Theta_D) \rightarrow H^1(V_0, \Theta_2) \rightarrow H^1(X, \mathcal{O}_X) \otimes H^0(D, \Theta_D) \rightarrow 0.$$

But we know that  $H^0(D, \Theta_D)$  is the Lie algebra  $\mathfrak{a}$  of the group

$$A = \mathrm{GL}(2, \mathbb{C}) / \mathbb{C}^* = \mathrm{SL}(2, \mathbb{C}) / \{\pm 1\}$$

of automorphisms of  $D$ , and that  $H^1(D, \Theta_D) = 0$ , as we can easily see by taking a cover of  $D$  by two open subsets. We have already seen (in [Talk no. 1]) that, if  $X = E/\Gamma$ , then  $H^1(X, \mathcal{O}) = \mathrm{Hom}(\Gamma, \mathbb{C}) / \mathrm{Hom}_{\mathbb{C}}(E, \mathbb{C})$  is of dimension 2. So  $H^1(V_0, \Theta_2) = H^1(X, \mathcal{O}) \otimes \mathfrak{a}$  is of dimension 6. The cup product

$$H^1(V_0, \Theta_2) \otimes H^1(V_0, \Theta_2) \rightarrow H^2(V_0, \Theta_2)$$

is given by the formula

$$[(\gamma \otimes \alpha) \smile (\gamma' \otimes \alpha')] = (\gamma \smile \gamma') \otimes [\alpha, \alpha'].$$

The cone of elements  $\varphi \in H^1(V_0, \Theta_2)$  such that  $[\varphi \smile \varphi] = 0$  can be identified with the cone of rank 1 tensors in  $H^1(X, \mathcal{O}) \otimes \mathfrak{a}$ . Indeed, if  $\varphi = \gamma \otimes \alpha$ , then

$$[\varphi \smile \varphi] = (\gamma \smile \gamma) \otimes [\alpha, \alpha] = 0 \otimes 0 = 0$$

and, if  $\varphi$  is not a simple tensor, then we have

$$\varphi = \gamma \otimes \alpha + \gamma' \otimes \alpha'$$

with  $\gamma$  and  $\gamma'$  independent, and  $\alpha$  and  $\alpha'$  independent, so

$$[\varphi \smile \varphi] = 2(\gamma \smile \gamma') \otimes [\alpha, \alpha'] \neq 0.$$

## 2 The mixed space $V$

In this example, every element of  $H^1(V_0, \Theta_2)$  whose primary obstruction is zero is a deformation vector. More precisely:

| p. 2-07

**Proposition 2.** *There exists a mixed space  $\pi: V \rightarrow B$  and a point  $b_0 \in B$  such that*

1.  $\pi^{-1}(b_0) = V_0$  (the manifold defined in III.1) ;
2. there exists an isomorphism  $\sigma$  from a  $\mathbb{C}$ -analytic space  $B$  to the cone of elements  $\varphi \in H^1(V_0, \Theta_2)$  such that  $[\varphi \smile \varphi] = 0$  ; and

3. for every subspace  $B'$  of  $B$  that has no singularities at  $b_0$ , the Spencer–Kodaira map  $\rho$  from the tangent space of  $B'$  at  $b_0$  to  $H^1(V_0, \Theta)$  agrees with  $\sigma: B' \rightarrow H^1(V_0, \Theta_2)$ .

Let  $H$  be the analytic space of homomorphisms from  $\Gamma$  to  $\mathfrak{a}$  whose images are contained in a vector subspace of  $\mathfrak{a}$  that is 1-dimensional over  $\mathbb{C}$  (i.e.  $(4 \times 2)$  matrices of rank 1 with coefficients in  $\mathbb{C}$ ). For every  $h \in H$ ,  $e \circ h$  is a homomorphism from  $\Gamma$  to  $A$ , where  $e: \mathfrak{a} \rightarrow A$  denotes the exponential map, and we construct a manifold  $V_h$  that is fibred over  $X$  with fibre  $D$  as follows:  $V_h$  is the quotient of  $E \times D$  by the equivalence relation defined by  $\Gamma$  acting via

$$\gamma \star (x, y) = (x + \gamma, ((e \circ h)(\gamma)) \cdot y).$$

These manifolds are the fibres of a mixed space  $W \rightarrow H$ , where  $W$  is the quotient of  $H \times E \times D$  by the equivalence relation defined by  $\Gamma$  acting via

$$\gamma \star (h, x, y) = (h, x + \gamma, (e \circ h(\gamma)) \cdot y).$$

We now place the following equivalence relation on  $H$ : we have  $h' \sim h$  if and only if  $(h' - h)$  extends to an  $\mathbb{C}$ -linear map  $f: E \rightarrow \mathfrak{a}$ . Note that, if  $h'(\Gamma)$  and  $h(\Gamma)$  are contained in the same subspace  $L$  of  $\mathfrak{a}$  of dimension 1 over  $\mathbb{C}$  (or if  $h' \sim h$ ), then we also have  $f(E) \subset L$  (or  $h \sim 0$  and  $h' \sim 0$ ). In both cases,  $V_h$  and  $V_{h'}$  are isomorphic, and we have an isomorphism  $i_{h',h}: V_h \rightarrow V_{h'}$  defined by

$$i_{h',h}(x, y) = (x, e \circ f(x) \cdot y)$$

(in the first case), or

$$i_{h',h} = i_{h',0} \circ i_{0,h}$$

(in the second case). If  $h, h'$ , and  $h''$  are in the same class, then we have  $i_{h''h} = i_{h''h'} \circ i_{h'h}$ , and we can place on  $W$  the equivalence relation | p. 4-08

$$(h', z') \sim (h, z) \iff h' \sim h \text{ or } z' = i_{h'h}z$$

for  $h, h' \in H$ ,  $z \in V_h$ , and  $z' \in V_{h'}$ .

Let  $B$  and  $V$  be the quotients of  $H$  and  $W$  (respectively) by these equivalence relations. We have a projection  $V \rightarrow B$ . To show that the structures of a  $\mathbb{C}$ -analytic space on  $H$  and  $W$  induce structures of a  $\mathbb{C}$ -analytic space on their quotients  $B$  and  $V$ , it suffices to remark that we can lift  $B$  to a analytic subspace of  $H$ : let, for example,  $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$  be a basis of  $\Gamma$  such that  $(\gamma_1, \gamma_2)$  is a basis of  $E$  over  $\mathbb{C}$ ; then each class  $b \in B$  contains exactly one element  $h \in H$  such that

$$h(\gamma_1) = h(\gamma_2) = 0.$$

### 3 Calculating $\rho_0$

Let  $T$  be the Zariski tangent space of  $B$  at  $b_0$ , i.e. the dual of  $\mathcal{J}/\mathcal{J}^2$ , where  $\mathcal{J}$  is the ideal of germs at  $b_0$  of analytic functions on  $B$  that are zero at  $b_0$ . Then  $T_0$  can be identified with  $\text{Hom}(\Gamma, \mathfrak{a})/\text{Hom}_{\mathbb{C}}(E, \mathfrak{a})$ . Also,

$$\begin{aligned} H^1(V_0, \Theta) &= H^1(V_0; \Theta_1) \oplus H^1(V_0; \Theta_2) \\ &= (H^1(X; \mathcal{O}) \otimes E) \oplus (H^1(X; \mathcal{O}) \otimes \mathfrak{a}), \end{aligned}$$

and the second term of this term can be identified with the quotient  $\text{Hom}(\Gamma, \alpha)/\text{Hom}_{\mathbb{C}}(E, \alpha)$ . We are going to show that the map  $\rho_0: T_0 \rightarrow H^1(V_0; \Theta)$  is exactly the canonical injection defined by these identifications.

Let  $u \in T_0 = \text{Hom}(\Gamma, \alpha)/\text{Hom}(E, \alpha)$  be the class of an element  $h \in \text{Hom}(\Gamma, \alpha)$ , which we suppose to be of rank 1. Then we can write  $h$  in the form  $\eta \otimes \sigma$ , where  $\eta \in \text{Hom}(\Gamma, \mathbb{C})$ ,  $\sigma \in \alpha$ , and we can consider  $h$  as a tangent vector to  $H$  at 0. Let  $\bar{h}$  be the field of tangent vectors to  $H \times E \times D$  at  $0 \times E \times D$  that projects onto  $h$ , and thus whose components over  $E \times D$  are zero. Let  $(U_i)$  be a cover of  $X = E/\Gamma$  by simply connected open subsets, and choose, for each  $i$ , a component  $\tilde{U}_i$  of the inverse image of  $U_i$  in  $E$ . We will denote by  $v_i$  the image over  $U_i \times D$  of the field  $\bar{h}|_{\tilde{U}_i \times D}$ . This is a projectable holomorphic field on  $0 \times U_i \times D$  of tangent vectors of  $H \times U_i \times D$ , and we set  $w_{ij} = v_j - v_i$ , so that  $w_{ij}$  is a vertical holomorphic field on  $U_{ij} \times D$ , and these fields form a cocycle whose cohomology class will be, by definition,  $\rho_0(u)$ . | p. 4-09

Let  $x \in U_{ij}$ , and let  $\tilde{x}_i$  and  $\tilde{x}_j$  be its inverse image in  $\tilde{U}_i$  and  $\tilde{U}_j$  (respectively). We have that  $\tilde{x}_j = \tilde{x}_i + \gamma_{ij}(x)$ , where  $\gamma_{ij}(x) \in \Gamma$ , and

$$w_{ij}(x) = \bar{h}(\tilde{x}_j) - [\gamma_{ij}(x)]_*(\bar{h}(\tilde{x}_i)) = -h(\gamma_{ij}(x)) \in \alpha.$$

Now  $w_{ij}$  is a vector field on  $D$ , and so

$$(w_{ij}) \in Z^1(V_0, (U_i \times D); \Theta_2),$$

and  $w_{ij}$  is of the form  $\zeta \otimes \alpha$ , where  $\zeta \in Z^1(V_0, (U_i \times D); \mathcal{O})$  is the cocycle defined by  $\zeta_{ij}(x) = -\eta(\gamma_{ij}(x))$ . This is a cocycle whose cohomology class is (up to a sign) the element of  $H^1(V_0, \mathcal{O})$  that is identified with the class  $\eta$  in  $\text{Hom}(\Gamma, \mathbb{C})/\text{Hom}_{\mathbb{C}}(E, \mathbb{C})$ . QED.

## Appendix: Higher obstructions

### I Definition of obstructions

#### 1 The sheaf of germs of vertical automorphisms

Let  $V_0$  be a  $\mathbb{C}$ -analytic manifold, which we assume to be compact, and  $B$  a  $\mathbb{C}$ -analytic space, and let  $b_0 \in B$ . We are going to define a sheaf  $\Gamma$  of non-abelian groups on  $V_0$ . For every open subset  $U$  of  $V_0$ , consider the isomorphisms of analytic varieties  $\gamma: W \rightarrow W'$ , where  $W$  and  $W'$  are open subsets of  $B \times V_0$  that contain  $\{b_0\} \times U$ , such that the following conditions are satisfied: | p. 4-10

### References

- [1] Grothendieck, A. *A general theory of fibre spaces with structure sheaf*. University of Kansas, Department of Mathematics (1955).
- [2] Haefliger, A. Structures feuilletées et cohomologie à valeur dans un faisceau de groupoides. *Comm. Math. Helvet.* **32** (1957/58), 248–239.

- [3] Kodaira, K. and Spencer, D. On deformation of complex analytic structures, I. *Annals of Math.* **67** (1958), 328–401.
- [4] Kodaira, K. and Nirenberg, L. and Spencer, D.C. On the existence of deformations of complex analytic structures. *Annals of Math.* **68** (1958), 450–459.
- [5] Kuranishi, M. On the locally complete families of complex analytic structures. (To appear in the Annals of Mathematics).